

A FINITE ELEMENT METHOD FOR HIGH-CONTRAST INTERFACE PROBLEMS WITH ERROR ESTIMATES INDEPENDENT OF CONTRAST

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ABSTRACT. We define a new finite element method for a steady state elliptic problem with discontinuous diffusion coefficients where the meshes are not aligned with the interface. We prove optimal error estimates in the L^2 norm and H^1 weighted semi-norm independent of the contrast between the coefficients. Numerical experiments validating our theoretical findings are provided.

1. INTRODUCTION

In this article we develop a finite element method for a steady state interface problem. We pay particular attention to high-contrast problems, proving optimal error estimates independent of the contrast of the discontinuous constant coefficients for the numerical method.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with an immersed interface Γ such that $\bar{\Omega} = \bar{\Omega}^- \cup \bar{\Omega}^+$ with $\Omega^- \cap \Omega^+ = \emptyset$, and $\Gamma = \bar{\Omega}^- \cap \bar{\Omega}^+$. We assume that Γ does not intersect $\partial\Omega$, enclosing either Ω^- or Ω^+ . Our numerical method will approximate a solution of the problem below.

$$(1.1a) \quad -\rho^\pm \Delta u^\pm = f^\pm \quad \text{in } \Omega^\pm,$$

$$(1.1b) \quad u = 0 \quad \text{on } \partial\Omega,$$

$$(1.1c) \quad [u] = 0 \quad \text{on } \Gamma,$$

$$(1.1d) \quad [\rho D_{\mathbf{n}} u] = 0 \quad \text{on } \Gamma.$$

The jumps across the interface Γ are defined as

$$[\rho D_{\mathbf{n}} u] = \rho^- D_{\mathbf{n}^-} u^- + \rho^+ D_{\mathbf{n}^+} u^+ = \rho^- \nabla u^- \cdot \mathbf{n}^- + \rho^+ \nabla u^+ \cdot \mathbf{n}^+, \quad [u] = u^+ - u^-,$$

where $u^\pm \equiv u|_{\Omega^\pm}$ and \mathbf{n}^\pm is the unit outward normal to Ω^\pm . We furthermore assume that $\rho^+ \geq \rho^- > 0$ are constants and that the interface Γ is a closed, simple and regular \mathcal{C}^2 curve with an arc-length parameterization \mathbf{X} .

There has been a recent surge in the development of finite element methods for interface problems. See for instance [21, 24, 3, 7, 6, 25, 5, 16, 15, 18, 12, 22, 2, 23, 1, 8], to name a few. Among the articles where the discretization is based on meshes not aligned with the interface, most of the methods focus on low contrast problems and only a few address the high contrast problems ($\rho^+/\rho^- \gg 1$). For example, Burman et al. [7] introduced an unfitted Nitsche's method with averages and stabilization techniques for arbitrarily high-contrast problems, presenting bounds for the condition number of the stiffness matrix, although a rigorous error analysis was not given in that paper. Another example, is given by Chu et al. [8] that uses multiscale techniques to build basis functions, an approach that seems well suited for high curvature problems (e.g. inclusions completely contained in a triangle). However, in regions where the curvature of the

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interface is small it appears that their main a priori estimate degenerates (see Theorem 3.9 in [8]), forcing them to refine the mesh on those regions in order to be aligned with the interface. In our approach we do not need mesh refinements to make the triangulations aligned with the interface, however, we do not address high curvature problems.

In order to put our contribution in context, let us explain two popular finite element approaches for problem (1.1). The first approach is to double the degrees of freedom on triangles that intersect the interface and then add penalty terms to weakly enforce the continuity across the interface, see for example [7]. Burman et al. [7] demonstrated that in addition to penalizing the jumps across Γ it is necessary to add a flux stabilization term. This method is the so-called stabilized unfitted Nitsche's method. The stabilization term penalizes the jumps of the gradient on edges that belong to triangles that intersect the interface. As Burman et al. [7] showed, in order to obtain a method that is robust with respect to diffusion contrast and robust with respect to the way Γ cuts triangles, this type of penalization is necessary. The second common approach, and the one we focus in this paper, is to define local piecewise polynomial finite element spaces on triangles that intersect the interface Γ (see for instance [1, 13, 12, 8, 18, 17]). The basis functions are constructed by having them satisfy the continuity of the solution and the continuity of the flux strongly across Γ . Unlike the unfitted Nitsche's method that weakly imposes the interface conditions, in this approach the flux conservation and the continuity of the solution are enforced strongly, for example at certain points on Γ , without requiring stabilization terms on Γ . This is an important and distinguishable feature of the Immersed Interface Method using a Finite Element formulation (Immersed Finite Element Methods, see [20]). These basis functions are defined locally on each triangle, and therefore they are naturally discontinuous across edges of the triangulation. Namely, Adjerid et al. in [1] proposed to penalize jumps of the trial functions across the edges. Similarly, Lin et al. in [21] added similar penalty terms and proved optimal error estimates. However, in their analysis they do not consider high-contrast problems.

In this paper we follow this approach, defining local basis functions that are piecewise polynomials on each side of triangles that are cut by Γ . However, an additional stabilization term is added as compared to the methods of Adjerid et al. [1] and Lin et al. [21], allowing us to prove error estimates that are independent of the contrast ρ^+/ρ^- . By stabilization term we refer to a penalization of the jumps of the normal derivatives of the approximation across the edges that belong to triangles intersecting Γ . This idea was used before by Burman et al. [7], however here we use different stabilization parameters (and of course different basis functions). Roughly speaking, the reason this flux stabilization is important for high contrast problems, is that one does not want to move estimates from Ω^+ to Ω^- because ρ^+ could be much larger than ρ^- . However, triangles that are cut by Γ might have a thin part in Ω^+ and therefore inverse estimates might be affected. By adding the jumps of derivatives we can transfer the estimates to a neighboring triangle that will have a larger portion in Ω^+ .

In order to prove error estimates independent of the contrast ρ^+/ρ^- we assume H^2 regularity of u on both Ω^+ and Ω^- . To be more precise, the error estimate presented in this paper for the energy norm (see (4.1)) is

$$(1.2) \quad \|u - u_h\|_V \leq Ch \left\{ \sqrt{\rho^-} (\|Du\|_{L^2(\Omega^-)} + \|D^2u\|_{L^2(\Omega^-)}) + \sqrt{\rho^+} (\|Du\|_{L^2(\Omega^+)} + \|D^2u\|_{L^2(\Omega^+)}) \right\}.$$

Consequently, assuming that Ω is convex and using regularity estimates (see [8]) we will have the result

$$(1.3) \quad \|u - u_h\|_V \leq C \frac{h}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}.$$

In addition, using a duality argument we prove the estimate $\|u - u_h\|_{L^2(\Omega)} \leq C(h^2/\rho^-)\|f\|_{L^2(\Omega)}$. The constants in the estimates depend on the geometry, including the curvature of Γ .

The outline of the paper is as follows. We formulate the discrete problem as finding $u_h \in V_h$ such that $a_h(u_h, v_h) = (f, v_h)$, $\forall v_h \in V_h$. The discrete space V_h is introduced in Section 2.1 and the bilinear form $a_h(\cdot, \cdot)$ in Section 2.2. In Section 3, fundamental results on element-wise weighted L^2 and H^1 norm approximation for the space V_h are established. Coercivity and continuity of the bilinear form a_h are studied in Section 4.1 and Section 4.2, respectively. In particular for the continuity, we note that the use of an augmented norm is necessary for the analysis due to the presence of the penalty terms involving flux jumps. In Section 5 the bound (1.2) is established by estimating the approximation error and the consistency error across Γ and across elements near Γ . The error estimate in the L^2 norm is also proved in this section. Section 6 is devoted to present extensions of the method to three dimensions, discuss related methods, and state some concluding remarks. Finally, in Section 7 we provide numerical experiments corroborating our theoretical findings. An Appendix containing technical proofs and a computational consideration is also included.

2. THE FINITE ELEMENT METHOD

2.1. Notation and local finite element space. In this section we present a finite element method for problem (1.1) using piecewise linear polynomials.

We next develop notation. Let \mathcal{T}_h , $0 < h < 1$ be an admissible family of triangulations of Ω (conforming), with $\bar{\Omega} = \cup_{T \in \mathcal{T}_h} \bar{T}$ and the elements T are mutually disjoint. Let h_T denote the diameter of the element T and $h = \max_T h_T$. We let \mathcal{E}^h be the set of all edges of the triangulation. We adopt the convention that edges e , elements T , sub-edges $e^\pm := e \cap \Omega^\pm$, sub-elements $T^\pm := T \cap \Omega^\pm$ and sub-regions Ω^\pm are open sets, and we use the over-line symbol to refer to their closure.

Let \mathcal{T}_h^Γ denotes the set of triangles $T \in \mathcal{T}_h$ such that T intersects Γ . We let \mathcal{E}_h^Γ be the set of all the three edges of triangles in \mathcal{T}_h^Γ . Since Γ is \mathcal{C}^2 we have that $\|\mathbf{X}''\|_{L^\infty} < \infty$ and we define the maximum curvature $\kappa := \|\mathbf{X}''\|_{L^\infty}$. Our analysis, will be valid when h is sufficiently small. To make this precise, we will use the concept of a tubular neighborhood whose existence is a standard result in differential geometry; see [9] Section 2-7, Proposition 3.

Lemma 1. (*Existence of r -tubular neighborhood*) *Let Γ be a regular, simple, \mathcal{C}^2 curve. For every $x \in \Gamma$ consider the line segment $N_x(r)$ of length $2r$ centered at x and perpendicular to Γ at x . Define the tubular neighborhood of radius r of Γ by $\text{Tub}(r) = \cup_{x \in \Gamma} N_x(r)$. Then, there exists $r > 0$ such that for any two points $x, y \in \Gamma$, $x \neq y$, the line segments $N_x(r)$ and $N_y(r)$ are disjoint.*

From now on we will work under the following assumptions.

Assumption. *Given Lemma 1 we make the following assumptions on the triangulation:*

- (1) *We assume that the triangulation is shape-regular, see [4].*
- (2) *We assume $h < r/2$ where r is the radius of the tubular neighborhood of Γ .*
- (3) *The interface intersects the boundary of an element at most twice and at different edges.*

It is well known that $r \leq 1/\kappa$, and hence by our assumption $h < 1/(2\kappa)$. The radius r also bounds from below how close the curve Γ comes from self-intersecting (e.g. consider a dumbbell with a thin middle section). We make use of these assumptions to prove some of the technical lemmas (see Lemmas 6, 11, A.1).

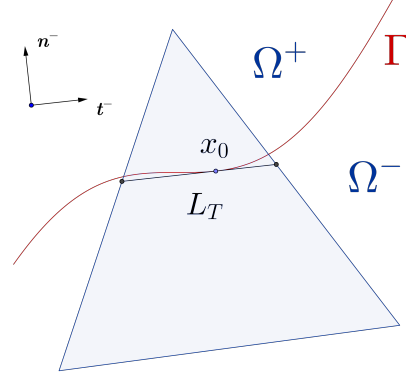
In addition, we define an element patch ω_T of a triangle T , its restriction to Ω^\pm and its intersection with Γ by

$$\omega_T := \text{Int}\left\{ \bigcup_{K \in \mathcal{T}_h} \overline{K} : \overline{K} \cap \overline{T} \neq \emptyset \right\}, \quad \omega_T^\pm := \omega_T \cap \Omega^\pm, \quad \omega_T^\Gamma := \omega_T \cap \Gamma,$$

where “Int” denotes the interior of the set.

The introduction of notation for the patch will be relevant in the proof of the interpolation error further forward. We first need to build our local finite element space (on each element). To this end, we let x_0 be the midpoint with respect to the arc-length on the curve segment $T_\Gamma := T \cap \Gamma$. We note that the midpoint choice is a preference of the authors, the proofs below hold for any $x_0 \in T_\Gamma$. Let L_T be the line segment inside T which is tangent to Γ at x_0 . We define by \mathbf{t}^\pm the unit tangent vector to Γ by a 90° clockwise rotation of \mathbf{n}^\pm . Figure 1 illustrates the definitions and notations introduced above.

FIGURE 1. Illustration of our notation on an element $T \in \mathcal{T}_h^\Gamma$.



In order to define our finite element space, we will need the following lemma.

Lemma 2. *Consider the operator $\Upsilon : \mathbb{P}^1(T^+) \rightarrow \mathbb{P}^1(T^-)$ defined by*

$$(2.1) \quad \Upsilon(v)(x_0) := v(x_0),$$

$$(2.2) \quad (D_{\mathbf{t}_0^+} \Upsilon(v))(x_0) := (D_{\mathbf{t}_0^+} v)(x_0),$$

$$(2.3) \quad \rho^-(D_{\mathbf{n}_0^+} \Upsilon(v))(x_0) := \rho^+(D_{\mathbf{n}_0^+} v)(x_0),$$

where $\mathbf{n}_0^\pm = \mathbf{n}^\pm(x_0)$ and $\mathbf{t}_0^\pm = \mathbf{t}^\pm(x_0)$. Then, Υ is well defined.

Note that the exact solution u^\pm satisfies the transmission conditions (2.1), (2.2) and (2.3) on all points over Γ .

Next, given $T \in \mathcal{T}_h^\Gamma$ and for each $v \in \mathbb{P}^1(T^+)$ we can consider the unique corresponding function

$$G(v) = \begin{cases} v, & \text{on } T^+, \\ \Upsilon(v), & \text{on } T^-. \end{cases}$$

Let $\text{span}\{v_1, v_2, v_3\}$ be a basis for $\mathbb{P}^1(T)$ restricted to T^+ . Then we define the local finite element space

$$(2.4) \quad S^1(T) = \begin{cases} \text{span}\{G(v_1), G(v_2), G(v_3)\}, & \text{if } T \in \mathcal{T}_h^\Gamma, \\ \mathbb{P}^1(T), & \text{if } T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma. \end{cases}$$

An explicit construction of basis functions of the local space $S^1(T)$ is given in Appendix B. The global finite element space is defined by

$$V_h := \{v : v|_T \in S^1(T), \forall T \in \mathcal{T}_h, v \text{ is continuous across all edges in } \mathcal{E}_h \setminus \mathcal{E}_h^\Gamma\}.$$

2.2. Finite element method. We begin this section by introducing some standard discontinuous finite element notation for jumps and averages.

For a piecewise smooth function v with support on \mathcal{T}_h , we define its average and jump across an interior edge $e \in \mathcal{E}_h \setminus \partial\Omega$, shared by elements T_1 and T_2 , as

$$\{v\} = \frac{v|_{T_1} + v|_{T_2}}{2}, \quad \llbracket v \rrbracket = v|_{T_1} \mathbf{n}_1 + v|_{T_2} \mathbf{n}_2,$$

where \mathbf{n}_1 and \mathbf{n}_2 are the outward pointing unit normal vectors to T_1 and T_2 , respectively. Similarly, if $\boldsymbol{\tau}$ is a vector-valued function, piecewise smooth on \mathcal{T}_h , its average and normal jump across an interior edge e are defined as

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}|_{T_1} + \boldsymbol{\tau}|_{T_2}}{2}, \quad \llbracket \boldsymbol{\tau} \rrbracket = \boldsymbol{\tau}|_{T_1} \cdot \mathbf{n}_1 + \boldsymbol{\tau}|_{T_2} \cdot \mathbf{n}_2.$$

Next we introduce the finite element approximation to problem (1.1). First, we define the space V as the union of the broken Sobolev spaces

$$(2.5) \quad V := H_h^2(\Omega^+) \cup H_h^2(\Omega^-), \quad \text{where} \quad H_h^2(\Omega^\pm) = \{v : v|_{T^\pm} \in H^2(T^\pm), \text{ for all } T \in \mathcal{T}_h\}.$$

We can then define the bilinear form $a_h : V \times V \rightarrow \mathbb{R}$ and the linear functional $(f, \cdot) : V \rightarrow \mathbb{R}$ by

$$(2.6) \quad \begin{aligned} a_h(w, v) &:= \int_{\Omega} \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^\Gamma} \int_e (\{\rho \nabla_h v\} \cdot \llbracket w \rrbracket + \{\rho \nabla_h w\} \cdot \llbracket v \rrbracket) \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\gamma}{|e^-|} \int_{e^-} \rho^- \llbracket w \rrbracket \cdot \llbracket v \rrbracket + \frac{\gamma}{|e^+|} \int_{e^+} \rho^+ \llbracket w \rrbracket \cdot \llbracket v \rrbracket \right) \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \left(|e^-| \int_{e^-} \rho^- \llbracket \nabla_h v \rrbracket \llbracket \nabla_h w \rrbracket + |e^+| \int_{e^+} \rho^+ \llbracket \nabla_h v \rrbracket \llbracket \nabla_h w \rrbracket \right), \\ (f, v) &:= \sum_{T \in \mathcal{T}_h} \left(\int_{T^-} f^- v + \int_{T^+} f^+ v \right), \end{aligned}$$

for a penalty parameter $\gamma > 0$. The discrete gradient operator ∇_h is piecewise defined on T^\pm for an element $T \in \mathcal{T}_h$ by

$$\nabla_h v|_{T^\pm} = Dv = \nabla v.$$

Finally, the finite element approximation solves: Find $u_h \in V_h$ such that

$$(2.7) \quad a_h(u_h, v) = (f, v), \quad \text{for all } v \in V_h.$$

Note that in (2.6) we not only penalize the jumps of the function but also the normal jumps of the first derivatives across edges. This will allow us to prove coerciveness and a priori error estimates independent of the contrast of the coefficients and also independent of how small T^+

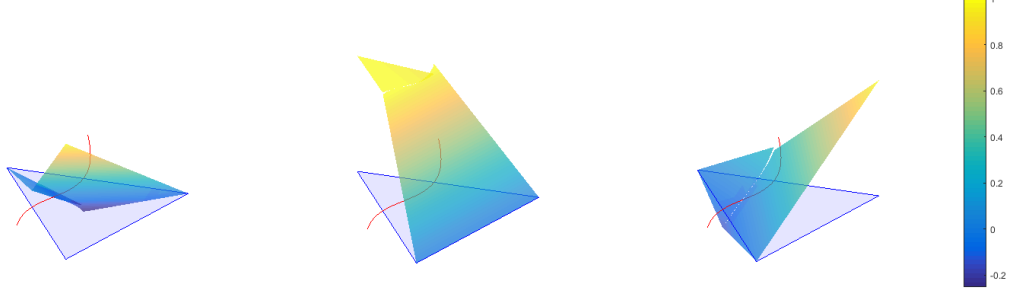


FIGURE 2. Illustration of basis functions on triangle in Figure 1 with $\rho^+ = 100$ and $\rho^- = 1$.

or T^- might be. Finally, we like to stress that in (2.6) only the normal derivative jumps are penalized, not the tangential derivative jumps.

3. LOCAL APPROXIMATION

In this section we show that our finite element space has optimal local approximation properties. Henceforth we will keep track, as much as possible, on how constants depend on the maximum curvature κ , and in general in the geometry of sub-domains. When we use standard results (e.g. regularity and extensions lemmas) we just state that the constants depend on the geometry without explicitly saying how they scale with quantities such as curvature and the radius of the tubular neighborhood r . In order to define an interpolation operator onto V_h we first state an extension result. Consider $u^\pm \in H^2(\Omega^\pm)$ (with $u^\pm \equiv 0$ on $\partial\Omega^\pm \cap \partial\Omega$). Then, there exists a constant C and extensions $u_E^\pm \in H^2(\Omega)$ with the following properties

$$(3.1) \quad u_E^\pm = u^\pm \quad \text{in } \Omega^\pm,$$

$$(3.2) \quad \|Du_E^\pm\|_{L^2(\Omega)} + \|D^2u_E^\pm\|_{L^2(\Omega)} \leq C(\|Du^\pm\|_{L^2(\Omega^\pm)} + \|D^2u^\pm\|_{L^2(\Omega^\pm)}).$$

This result follows from Theorem 7.25 in [11] and applying Poincaré's inequalities. Considering that we will only require the extensions u_E^\pm to be defined on patches of elements on \mathcal{T}_h^Γ , in fact, we only need the following more *local* bound which could lead to a better geometric constant:

$$(3.3) \quad \|Du_E^\pm\|_{L^2(\Omega^\pm \cup T_{ub}(2h))} + \|D^2u_E^\pm\|_{L^2(\Omega^\pm \cup T_{ub}(2h))} \leq C_E(\|Du^\pm\|_{L^2(\Omega^\pm)} + \|D^2u^\pm\|_{L^2(\Omega^\pm)}).$$

For the sake of simplicity, we do not prove here how C_E depends on the geometric constants (e.g. κ and r).

Definition 3.1. Let $u^\pm \in H^2(\Omega^\pm)$. For each $T \in \mathcal{T}_h^\Gamma$ we define $I_T u \in S^1(T)$. In fact, we define $I_T u$ on all ω_T

$$I_T u = \begin{cases} I_T^+ u, & \text{in } \omega_T^+ \\ I_T^- u, & \text{in } \omega_T^-, \end{cases}$$

where I_T^\pm are defined satisfying the following conditions

$$(3.4) \quad \begin{cases} (I_T^- u)(x_0) &:= (J_T u_E^+)(x_0) &=: (I_T^+ u)(x_0) \\ (D_{\mathbf{t}_0^+} I_T^- u)(x_0) &:= (D_{\mathbf{t}_0^+} (J_T u_E^+))(x_0) &=: (D_{\mathbf{t}_0^+} I_T^+ u)(x_0) \\ \rho^-(D_{\mathbf{n}_0^+} I_T^- u)(x_0) &:= \rho^-(D_{\mathbf{n}_0^+} J_T u_E^+)(x_0) &=: \rho^+(D_{\mathbf{n}_0^+} I_T^+ u)(x_0). \end{cases}$$

Here J_T is the L^2 projection operator onto $\mathbb{P}^1(\omega_T)$ (note that $\omega_T \subset Tub(r)$ since we are assuming $2h \leq r$).

If T does not belong to \mathcal{T}_h^Γ , we consider the sets $\Omega_h^\pm = \cup\{T \in \mathcal{T}_h : T \subset \Omega^\pm\}$, and we define $I_T u = I_{sz} u^\pm$, for $T \subset \Omega^\pm$, where $I_{sz} u^\pm$ is the Scott-Zhang interpolant of u^\pm defined on Ω_h^\pm . However, we need to modify the interpolant at every vertex x (if one exists) that is an endpoint of an edge e satisfying $e \subset \partial\Omega_h^+ \cap \partial\Omega_h^-$ (e.g. $e \subset \Gamma$). In this case, we define $I_{sz} u^\pm(x)$ to be the average of u^\pm on the edge e . Note that such an x might correspond to two edges. Either edge will work for the definition of $I_{sz} u^\pm(x)$. This will give $I_{sz} u^+ = I_{sz} u^-$ for every edge $e \subset \partial\Omega_h^+ \cap \partial\Omega_h^-$ since $u^+ = u^-$ on such an edge.

Consequently, we define the interpolation operator I_h onto the finite element space V_h as the restriction of the local interpolation operator I_T , i.e.

$$(3.5) \quad I_h u|_T = I_T(u)|_T, \quad \text{for all } T \in \mathcal{T}_h.$$

The local interpolant $I_T u$ was constructed so that Lemma 5 holds which in turn is the main tool to prove the crucial Lemma 10. The interpolator $I_T u$ was designed so that if we multiply the whole inequalities (3.6) and (3.7) by ρ^- and ρ^+ , respectively, only terms of the form $\rho^\pm \|D^j u_E^\pm\|_{L^2(\omega_T)}$ (after using $\rho^- \leq \rho^+$) will appear.

Since the proof of Lemma 5 is quite involved, we first prove a local energy stability of the interpolant whos proof is much easier and that motivates the definition of I_h . More specifically, we prove the following lemma.

Lemma 3. *It holds,*

$$\rho^+ \|\nabla(I_T^+ u)\|_{L^2(\omega_T^+)} + \rho^- \|\nabla(I_T^- u)\|_{L^2(\omega_T^-)} \leq C(\rho^- \|\nabla u_E^-\|_{L^2(\omega_T)} + \rho^+ \|\nabla u_E^+\|_{L^2(\omega_T)}).$$

where C is independent of ρ^\pm .

Proof. Clearly we have

$$\|\nabla(I_T^+ u)\|_{L^2(\omega_T^+)}^2 = \|D_{\mathbf{t}_0^+}(I_T^+ u)\|_{L^2(\omega_T^+)}^2 + \|D_{\mathbf{n}_0^+} I_T^+ u\|_{L^2(\omega_T^+)}^2.$$

Using that $D_{\mathbf{t}_0^+}(I_T^+ u)$ and $D_{\mathbf{n}_0^+}(I_T^+ u)$ are constant and using the definition of the interpolant (3.4) we have

$$\rho^+ \|\nabla(I_T u)\|_{L^2(\omega_T^+)} \leq C(\rho^+ \|D_{\mathbf{t}_0^+}(J_T u_E^+)\|_{L^2(\omega_T^+)} + \rho^- \|D_{\mathbf{n}_0^+}(J_T u_E^-)\|_{L^2(\omega_T^+)}).$$

We trivially have

$$\rho^+ \|\nabla(I_T u)\|_{L^2(\omega_T^+)} \leq C(\rho^+ \|\nabla(J_T u_E^+)\|_{L^2(\omega_T)} + \rho^- \|\nabla(J_T u_E^-)\|_{L^2(\omega_T)}).$$

Define $c^\pm = \frac{1}{|\omega_T|} \int_{\omega_T} u_E^\pm$ and note that $J_T c^\pm = c^\pm$. We then see

$$\rho^+ \|\nabla(I_T u)\|_{L^2(\omega_T^+)} \leq C(\rho^+ \|\nabla(J_T(u_E^+ - c^+))\|_{L^2(\omega_T)} + \rho^- \|\nabla(J_T(u_E^- - c^-))\|_{L^2(\omega_T)}).$$

Using an inverse estimate, the stability of the L^2 projection J_T , and Poincaré's inequality we get

$$\rho^+ \|\nabla(I_T u)\|_{L^2(\omega_T^+)} \leq C(\rho^+ \|\nabla u_E^+\|_{L^2(\omega_T)} + \rho^- \|\nabla u_E^-\|_{L^2(\omega_T)}).$$

In a similar way we can prove the estimate for $\rho^- \|\nabla(I_T^- u)\|_{L^2(\omega_T^-)}$ where we only need to use $\rho^- \leq \rho^+$.

□

Lemma 4. *Consider $T \in \mathcal{T}_h^\Gamma$. Then, for every $v \in \mathbb{P}^1(\omega_T)$ the following bound holds*

$$h_T^j \|D^j v\|_{L^2(\omega_T)} \leq C \left(h_T |v(x_0)| + h_T^2 |D_{\mathbf{n}_0^+} v(x_0)| + h_T^2 |D_{\mathbf{t}_0^+} v(x_0)| \right), \quad j = 0, 1.$$

Proof. Using the Cauchy-Schwarz inequality one has

$$h_T^j \|D^j v\|_{L^2(\omega_T)} \leq C h_T^{j+1} \|D^j v\|_{L^\infty(\omega_T)}, \quad j = 0, 1.$$

Using the fact that v is a linear function, the lemma follows. \square

We next prove a fundamental result of this paper, a local approximation property on the space $S^1(T)$.

Lemma 5. *Let $u^\pm \in H^2(\Omega^\pm)$ satisfying the interface conditions $[u] = 0$ and $[\rho D_{\mathbf{n}} u] = 0$. Then, for any $T \in \mathcal{T}_h^\Gamma$ and $j = 0, 1$, the following bounds hold:*

(3.6)

$$h_T^j \|D^j(u^- - I_T^- u)\|_{L^2(\omega_T^-)} \leq C(1 + \kappa) h_T^2 (\|Du_E^-\|_{L^2(\omega_T)} + \|D^2 u_E^-\|_{L^2(\omega_T)} + \|D^2 u_E^+\|_{L^2(\omega_T)})$$

and

$$(3.7) \quad h_T^j \|D^j(u^+ - I_T^+ u)\|_{L^2(\omega_T^+)} \leq C(1 + \kappa) h_T^2 (\|Du_E^+\|_{L^2(\omega_T)} + \|D^2 u_E^+\|_{L^2(\omega_T)} + \frac{\rho^-}{\rho^+} \|D^2 u_E^-\|_{L^2(\omega_T)}).$$

Proof. First note that by adding and subtracting $J_T u_E^\pm$ to $u_E^\pm - I_T^\pm u$, using the triangle inequality, and by means of the following well known approximation property for $\varphi \in H^2(\omega_T)$

$$(3.8) \quad h_T^j \|D^j(\varphi - J_T \varphi)\|_{L^2(\omega_T)} \leq C h_T^2 \|D^2 \varphi\|_{L^2(\omega_T)} \quad \text{for } j = 0, 1,$$

the proof of the lemma reduces to estimate

$$h_T^j \|D^j w_T^\pm\|_{L^2(\omega_T^\pm)}, \quad \text{where } w_T^\pm := J_T u_E^\pm - I_T^\pm u, \text{ for } w_T^\pm \in \mathbb{P}^1(\omega_T).$$

According to the definition of I_T (3.4), and denoting $\mathbf{n}_0^\pm = \mathbf{n}^\pm(x_0)$ and $\mathbf{t}_0^\pm = \mathbf{t}^\pm(x_0)$, we have

$$\begin{cases} w_T^-(x_0) &= J_T u_E^-(x_0) - J_T u_E^+(x_0), \\ (D_{\mathbf{t}_0^+} w_T^-)(x_0) &= (D_{\mathbf{t}_0^+} J_T u_E^-)(x_0) - (D_{\mathbf{t}_0^+} J_T u_E^+)(x_0), \\ (D_{\mathbf{n}_0^+} w_T^-)(x_0) &= 0, \end{cases}$$

and

$$\begin{cases} w_T^+(x_0) &= 0, \\ (D_{\mathbf{t}_0^+} w_T^+)(x_0) &= 0, \\ \rho^+(D_{\mathbf{n}_0^+} w_T^+)(x_0) &= \rho^+(D_{\mathbf{n}_0^+} J_T u_E^+)(x_0) - \rho^-(D_{\mathbf{n}_0^+} J_T u_E^-)(x_0). \end{cases}$$

Then, using Lemma 4 and the values derived for w_T^\pm we have

$$(3.9) \quad h_T^j \|D^j w_T^-\|_{L^2(\omega_T^-)} \leq C \left(h_T |w_T^-(x_0)| + h_T^2 |D_{\mathbf{t}_0^+} w_T^-(x_0)| \right),$$

and

$$(3.10) \quad h_T^j \|D^j w_T^+\|_{L^2(\omega_T^+)} \leq C h_T^2 |D_{\mathbf{n}_0^+} w_T^+(x_0)|.$$

We proceed by bounding the two terms in (3.9) and the term in (3.10), separately. For the first term in (3.9), we use that u is continuous on the interface (1.1c), in particular on ω_T^Γ , and we apply the triangle inequality

$$\begin{aligned} |w_T^-(x_0)| &= |(J_T u_E^- - J_T u_E^+)(x_0)| \leq \|J_T u_E^- - J_T u_E^+\|_{L^\infty(\omega_T^\Gamma)} \\ &\leq \|J_T u_E^- - u_E^-\|_{L^\infty(\omega_T^\Gamma)} + \|J_T u_E^+ - u_E^+\|_{L^\infty(\omega_T^\Gamma)}. \end{aligned}$$

Note that $ch_T \leq |\omega_T^\Gamma| \leq Ch_T$, where the second inequality follows from Lemma A.1. By means of a Sobolev inequality in one dimension, we have

$$\|J_T u_E^- - u_E^-\|_{L^\infty(\omega_T^\Gamma)} \leq C \left(\frac{1}{\sqrt{h_T}} \|J_T u_E^- - u_E^-\|_{L^2(\omega_T^\Gamma)} + \sqrt{h_T} \|D(J_T u_E^- - u_E^-)\|_{L^2(\omega_T^\Gamma)} \right),$$

consequently, using a trace inequality we obtain

$$\begin{aligned} \|J_T u_E^- - u_E^-\|_{L^\infty(\omega_T^\Gamma)} &\leq C (h_T^{-1} \|J_T u_E^- - u_E^-\|_{L^2(\omega_T)} + \|D(J_T u_E^- - u_E^-)\|_{L^2(\omega_T)}) \\ &\quad + C h_T \|D^2 u_E^-\|_{L^2(\omega_T)}. \end{aligned}$$

Hence, using the approximation property (3.8), we get

$$\|J_T u_E^- - u_E^-\|_{L^\infty(\omega_T^\Gamma)} \leq C h_T \|D^2 u_E^-\|_{L^2(\omega_T)}.$$

Analogously, we can show that

$$\|J_T u_E^+ - u_E^+\|_{L^\infty(\omega_T^\Gamma)} \leq C h_T \|D^2 u_E^+\|_{L^2(\omega_T)}.$$

Therefore, we have the bound for the first term in (3.9)

$$(3.11) \quad h_T |w_T^-(x_0)| \leq C h_T^2 (\|D^2 u_E^-\|_{L^2(\omega_T)} + \|D^2 u_E^+\|_{L^2(\omega_T)}).$$

We now turn to the second term in (3.9). We use the fact that $D_{\mathbf{t}_0^+} w_T^-$ is constant on ω_T^Γ to obtain

$$|D_{\mathbf{t}_0^+} w_T^-(x_0)| = |\omega_T^\Gamma|^{-1/2} \|D_{\mathbf{t}_0^+} w_T^-\|_{L^2(\omega_T^\Gamma)} \leq C h_T^{-1/2} \|D_{\mathbf{t}_0^+} (J_T u_E^- - J_T u_E^+)\|_{L^2(\omega_T^\Gamma)}.$$

Using the identities, denoting $\mathbf{t}^+ = \mathbf{t}^+(x)$ and $\mathbf{n}^+ = \mathbf{n}^+(x)$, for $x \in \omega_T^\Gamma$

$$(3.12) \quad D_{\mathbf{t}_0^+} u_E^\pm = (\mathbf{t}_0^+ \cdot \mathbf{t}^+) D_{\mathbf{t}^+} u_E^\pm + (\mathbf{t}_0^+ \cdot \mathbf{n}^+) D_{\mathbf{n}^+} u_E^\pm,$$

$$(3.13) \quad D_{\mathbf{t}^+} u_E^+ = D_{\mathbf{t}^+} u_E^-,$$

$$(3.14) \quad D_{\mathbf{n}^+} u_E^+ = \frac{\rho^-}{\rho^+} D_{\mathbf{n}^+} u_E^-,$$

we have

$$\begin{aligned} \|D_{\mathbf{t}_0^+} w_T^-\|_{L^2(\omega_T^\Gamma)} &= \|D_{\mathbf{t}_0^+} (J_T u_E^- - u_E^- + u_E^+ - J_T u_E^+) + (1 - \frac{\rho^-}{\rho^+}) (\mathbf{t}_0^+ \cdot \mathbf{n}^+) D_{\mathbf{n}^+} u_E^-\|_{L^2(\omega_T^\Gamma)} \\ &\leq \|D(J_T u_E^- - u_E^-)\|_{L^2(\omega_T^\Gamma)} + \|D(J_T u_E^+ - u_E^+)\|_{L^2(\omega_T^\Gamma)} \\ &\quad + (1 - \frac{\rho^-}{\rho^+}) \|(\mathbf{t}_0^+ \cdot \mathbf{n}^+) D u_E^-\|_{L^2(\omega_T^\Gamma)}. \end{aligned}$$

For the first two terms in the previous bound we use a trace inequality to obtain

$$\|D(J_T u_E^\pm - u_E^\pm)\|_{L^2(\omega_T^\Gamma)} \leq C \left(\frac{1}{\sqrt{h_T}} \|D(J_T u_E^\pm - u_E^\pm)\|_{L^2(\omega_T)} + \sqrt{h_T} \|D^2(J_T u_E^\pm - u_E^\pm)\|_{L^2(\omega_T)} \right),$$

and for the third term we use that $|\mathbf{t}_0^- \cdot \mathbf{n}^-| \leq \|\mathbf{X}''\|_{L^\infty} |\omega_T^\Gamma| = \mathcal{O}(\kappa h_T)$ on ω_T^Γ , $\rho^- \leq \rho^+$, and a trace inequality to obtain

$$(1 - \frac{\rho^-}{\rho^+}) \|(\mathbf{t}_0^+ \cdot \mathbf{n}^+) D u_E^-\|_{L^2(\omega_T^\Gamma)} \leq C \kappa \left(\sqrt{h_T} \|D u_E^-\|_{L^2(\omega_T)} + h_T^{3/2} \|D^2 u_E^-\|_{L^2(\omega_T)} \right).$$

Hence, applying approximation property (3.8) we obtain

$$(3.15) \quad h_T^2 |D_{\mathbf{t}_0^+} w_T^-(x_0)| \leq C(1 + \kappa) h_T^2 (\|D u_E^-\|_{L^2(\omega_T)} + \|D^2 u_E^-\|_{L^2(\omega_T)} + \|D^2 u_E^+\|_{L^2(\omega_T)}).$$

If we combine the inequalities (3.11) and (3.15) with (3.9), we arrive at the first result of the lemma, inequality (3.6).

Now we estimate the term in (3.10). We need to bound

$$|D_{\mathbf{n}_0^+} w_T^+(x_0)| = |\omega_T^\Gamma|^{-1/2} \|D_{\mathbf{n}_0^+} w_T^+\|_{L^2(\omega_T^\Gamma)} \leq Ch_T^{-1/2} \|D_{\mathbf{n}_0^+} (J_T u_E^+ - \frac{\rho^-}{\rho^+} J_T u_E^-)\|_{L^2(\omega_T^\Gamma)}.$$

Similarly to the previous bound, we note that

$$D_{\mathbf{n}_0^+} u_E^\pm = (\mathbf{n}_0^+ \cdot \mathbf{t}^+) D_{\mathbf{t}^+} u_E^\pm + (\mathbf{n}_0^+ \cdot \mathbf{n}^+) D_{\mathbf{n}^+} u_E^\pm,$$

and then using (3.13) and (3.14) we obtain

$$\|D_{\mathbf{n}_0^+} w_T^+\|_{L^2(\omega_T^\Gamma)} = \|D_{\mathbf{n}_0^+} (J_T u_E^+ - u_E^+ - \frac{\rho^-}{\rho^+} (J_T u_E^- - u_E^-)) + (1 - \frac{\rho^-}{\rho^+}) (\mathbf{n}_0^+ \cdot \mathbf{t}^+) D_{\mathbf{t}^+} u_E^+\|_{L^2(\omega_T^\Gamma)}.$$

The remaining of the proof is similar as above and we obtain

$$h_T^2 |D_{\mathbf{n}_0^+} w_T^+(x_0)| \leq C(1 + \kappa) h_T^2 \left(\frac{\rho^-}{\rho^+} \|D^2 u_E^-\|_{L^2(\omega_T)} + \|D u_E^+\|_{L^2(\omega_T)} + \|D^2 u_E^+\|_{L^2(\omega_T)} \right).$$

If we combine this inequality with (3.10), we arrive at our second result (3.7). \square

4. COERCIVITY AND CONTINUITY OF BILINEAR FORM

The aim of this section is to prove coercivity and continuity of the bilinear form $a_h(\cdot, \cdot)$ defined in (2.6).

4.1. Coercivity of Bilinear Form. We define the following energy norm $\|\cdot\|_V : V \rightarrow \mathbb{R}_0^+$

$$(4.1) \quad \begin{aligned} \|v\|_V^2 &= \|\sqrt{\rho} \nabla_h v\|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\rho^-}{|e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2 + \frac{\rho^+}{|e^+|} \|\llbracket v \rrbracket\|_{L^2(e^+)}^2 \right) \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\rho^- |e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \rho^+ |e^+| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^+)}^2 \right). \end{aligned}$$

In order to prove coercivity we will need the following lemma.

Lemma 6. *Let $e = \text{Int}(\partial T_1 \cap \partial T_2) \in \mathcal{E}_h^\Gamma$, for $T_1, T_2 \in \mathcal{T}^h$. Then, there exists a constant $\theta > 0$ such that*

$$|e^\pm|^2 \leq \theta \max_{i=1,2} |T_i^\pm|.$$

The constant θ depends on the shape regularity of the triangulation.

Proof. See Appendix A.1. \square

Lemma 7. *(Coercivity) If γ is large enough (depending on shape regularity of triangulation) there exists a constant $c > 0$, independent of h , ρ^- and ρ^+ , such that*

$$(4.2) \quad c \|v\|_V^2 \leq a_h(v, v), \quad \text{for all } v \in V_h.$$

Proof. Let $v \in V_h$, then

$$\begin{aligned} a_h(v, v) &= \|\sqrt{\rho} \nabla_h v\|_{L^2(\Omega)}^2 - 2 \sum_{e \in \mathcal{E}_h^\Gamma} \int_e \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\gamma}{|e^-|} \rho^- \|\llbracket v \rrbracket\|_{L^2(e^-)}^2 + \frac{\gamma}{|e^+|} \rho^+ \|\llbracket v \rrbracket\|_{L^2(e^+)}^2 \right) \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\rho^- |e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \rho^+ |e^+| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^+)}^2 \right). \end{aligned}$$

To prove the lemma, it is enough to bound the non-symmetric term

$$\int_e \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket = \int_{e^-} \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket + \int_{e^+} \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket.$$

Let $T_1, T_2 \in \mathcal{T}_h$ be such that $e = \text{Int}(\partial T_1 \cap \partial T_2)$ and set $T_e = T_1 \cup T_2$. Without loss of generality assume $|T_2^-| = \max_{i=1,2} |T_i^-|$. Then, according to Lemma 6

$$(4.3) \quad |e^-|^2 \leq \theta |T_2^-|.$$

Then, we see that

$$\begin{aligned} \int_{e^-} \{\rho \nabla v\} \cdot \llbracket v \rrbracket &= \int_{e^-} \frac{\rho^-}{2} \left((D_{\mathbf{n}_1} v) (v|_{T_1} - v|_{T_2}) + (D_{\mathbf{n}_2} v) (v|_{T_2} - v|_{T_1}) \right) \\ &= \int_{e^-} \rho^- (D_{\mathbf{n}_2} v) (v|_{T_2} - v|_{T_1}) + \frac{\rho^-}{2} (D_{\mathbf{n}_1} v + D_{\mathbf{n}_2} v) (v|_{T_1} - v|_{T_2}) \\ &= \int_{e^-} \rho^- (D_{\mathbf{n}_2} v) (v|_{T_2} - v|_{T_1}) + \frac{\rho^-}{2} (\llbracket \nabla_h v \rrbracket) (v|_{T_1} - v|_{T_2}). \end{aligned}$$

Using the fact that v is a linear function on T_2^- and using (4.3) we have

$$(4.4) \quad \left| 2 \int_{e^-} \rho^- (D_{\mathbf{n}_2} v) (v|_{T_2} - v|_{T_1}) \right| \leq C \|\rho^- Dv\|_{L^2(T_2^-)} \frac{1}{|e^-|^{1/2}} \|\llbracket v \rrbracket\|_{L^2(e^-)},$$

where C depends on θ . Therefore, we have

$$\left| 2 \int_{e^-} \rho^- (D_{\mathbf{n}_2} v) (v|_{T_2} - v|_{T_1}) \right| \leq \epsilon \rho^- \|\nabla_h v\|_{L^2(T_e^-)}^2 + \frac{\rho^- C^2}{\epsilon |e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2,$$

for any $\epsilon > 0$. Furthermore, we have

$$\left| \int_{e^-} \rho^- (\llbracket \nabla_h v \rrbracket) (v|_{T_1} - v|_{T_2}) \right| \leq \epsilon \rho^- |e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \frac{\rho^-}{\epsilon |e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2.$$

Collecting the last two estimates gives

$$\left| 2 \int_{e^-} \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket \right| \leq \epsilon \rho^- (|e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \|\nabla_h v\|_{L^2(T_e^-)}^2) + \frac{(C^2 + 1) \rho^-}{\epsilon |e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2.$$

Likewise, we can bound the integral over e^+ to get a combined result

$$\begin{aligned} \left| 2 \int_e \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket \right| &\leq \epsilon (\|\sqrt{\rho} \nabla_h v\|_{L^2(T_e)}^2 + \rho^- |e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \rho^+ |e^+| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^+)}^2) \\ &\quad + \frac{(C^2 + 1) \rho^-}{\epsilon |e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2 + \frac{(C^2 + 1) \rho^+}{\epsilon |e^+|} \|\llbracket v \rrbracket\|_{L^2(e^+)}^2. \end{aligned}$$

Summing over all edges $e \in \mathcal{E}_h^\Gamma$ we get

$$\begin{aligned} |2 \sum_{e \in \mathcal{E}_h^\Gamma} \int_e \{\rho \nabla_h v\} \cdot \llbracket v \rrbracket| &\leq \sum_{e \in \mathcal{E}_h^\Gamma} \epsilon (\|\sqrt{\rho} \nabla_h v\|_{L^2(T_e)}^2 + \rho^- |e^-| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^-)}^2 + \rho^+ |e^+| \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^+)}^2) \\ &\quad + \sum_{e \in \mathcal{E}_h^\Gamma} \frac{(C^2 + 1)\rho^-}{\epsilon |e^-|} \|\llbracket v \rrbracket\|_{L^2(e^-)}^2 + \frac{(C^2 + 1)\rho^+}{\epsilon |e^+|} \|\llbracket v \rrbracket\|_{L^2(e^+)}^2. \end{aligned}$$

Finally, the result follows by choosing $\epsilon = 1/2$ and then choosing $\gamma = \frac{(C^2+1)}{\epsilon} + 1$. \square

4.2. Continuity of bilinear form. Next we prove continuity of the bilinear form. To do this we define the augmented norm

$$(4.5) \quad \|v\|_W^2 = \|v\|_V^2 + \sum_{e \in \mathcal{E}_h^\Gamma} (\rho^- \|\{\nabla_h v\} \cdot \mathbf{n}\|_{L^2(e^-)}^2 |e^-| + \rho^+ \|\{\nabla_h v\} \cdot \mathbf{n}\|_{L^2(e^+)}^2 |e^+|).$$

Lemma 8. (Continuity) Suppose that $v, w \in V$. Then, there exists a constant $C > 0$, independent of h , ρ^- and ρ^+ , such that

$$a_h(w, v) \leq C \|w\|_W \|v\|_W.$$

Additionally, if $w \in V$ and $v \in V_h$ we have

$$(4.6) \quad a_h(w, v) \leq C \|w\|_W \|v\|_V.$$

Proof. We give a sketch of the proof by bounding each term of $a_h(\cdot, \cdot)$ in (2.6) separately. The first term can easily be bounded by Cauchy-Schwarz inequality

$$\int_{\Omega} \rho \nabla_h v \cdot \nabla_h w \leq \|\sqrt{\rho} \nabla_h v\|_{L^2(\Omega)} \|\sqrt{\rho} \nabla_h w\|_{L^2(\Omega)}.$$

The first part of the second term can be written as

$$\int_e \{\rho \nabla_h v\} \cdot \llbracket w \rrbracket = \int_{e^-} \{\rho^- \nabla_h v\} \cdot \llbracket w \rrbracket + \int_{e^+} \{\rho^+ \nabla_h v\} \cdot \llbracket w \rrbracket.$$

Using the Cauchy-Schwarz inequality one has

$$\int_{e^\pm} \{\rho^\pm \nabla_h v\} \cdot \llbracket w \rrbracket \leq \left(\sqrt{|e^\pm|} \sqrt{\rho^\pm} \|\{\nabla_h v\} \cdot \mathbf{n}\|_{L^2(e^\pm)} \right) \frac{\sqrt{\rho^\pm}}{\sqrt{|e^\pm|}} \|\llbracket w \rrbracket\|_{L^2(e^\pm)}.$$

Let $e = \text{Int}(\partial T_1 \cap \partial T_2)$. If $v \in V_h$ then $\nabla_h v|_{T_i^\pm}$ is constant for each $i = 1, 2$. Then, we can use Lemma 6 to get

$$\sqrt{|e^\pm|} \sqrt{\rho^\pm} \|\{\nabla_h v\} \cdot \mathbf{n}\|_{L^2(e^\pm)} \leq \sqrt{|e^\pm|} \sqrt{\rho^\pm} \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^\pm)} + C \sqrt{\rho^\pm} (\|\nabla_h v\|_{L^2(T_1^\pm)} + \|\nabla_h v\|_{L^2(T_2^\pm)})$$

where C here depends on θ . The third term can be bounded by

$$\int_{e^\pm} \rho^\pm \llbracket w \rrbracket \cdot \llbracket v \rrbracket \leq \rho^\pm \|\llbracket w \rrbracket\|_{L^2(e^\pm)} \|\llbracket v \rrbracket\|_{L^2(e^\pm)}.$$

Finally, the fourth term can be bounded by Cauchy-Schwarz inequality

$$\int_{e^\pm} \rho^\pm \llbracket \nabla_h w \rrbracket \cdot \llbracket \nabla_h v \rrbracket \leq \rho^\pm \|\llbracket \nabla_h w \rrbracket\|_{L^2(e^\pm)} \|\llbracket \nabla_h v \rrbracket\|_{L^2(e^\pm)}.$$

The proof is complete summing over the edges, using finite overlapping of the elements associated to the edges and using arithmetic-geometric mean inequality. \square

5. A PRIORI ERROR ESTIMATES

The purpose of this section is to prove a priori error estimates for the method defined in (2.7)-(2.6) in the energy and L^2 norm.

5.1. Energy error estimates.

Theorem 1. (*A priori energy error estimate*) Let u be the solution to problem (1.1) and $u_h \in V_h$ be its finite element approximation solution of (2.7). Then, there exists $C > 0$, independent of h , ρ^- and ρ^+ , such that

$$\|u - u_h\|_V \leq C h(1 + \kappa) \left(\sqrt{\rho^-} (\|Du\|_{L^2(\Omega^-)} + \|D^2u\|_{L^2(\Omega^-)}) + \sqrt{\rho^+} (\|Du\|_{L^2(\Omega^+)} + \|D^2u\|_{L^2(\Omega^+)}) \right).$$

The constant C depends on C_E from (3.3).

Proof. Note that $\|u - u_h\|_V \leq \|u - I_h u\|_V + \|I_h u - u_h\|_V$. Since $d_h := I_h u - u_h \in V_h$, coercivity of $a_h(\cdot, \cdot)$ (see (4.2)) gives

$$(5.1) \quad c \|d_h\|_V^2 \leq a_h(d_h, d_h) = a_h(I_h u - u, d_h) + a_h(u - u_h, d_h).$$

Using (4.6) we obtain, for $\epsilon > 0$

$$(5.2) \quad a_h(I_h u - u, d_h) \leq C \|I_h u - u\|_W \|d_h\|_V \leq C \left(\frac{\epsilon}{2} \|d_h\|_V^2 + \frac{1}{2\epsilon} \|I_h u - u\|_W^2 \right).$$

Choosing ϵ sufficiently small we have

$$(5.3) \quad \|d_h\|_V^2 \leq C \|I_h u - u\|_W^2 + a_h(u - u_h, d_h).$$

The proof of the theorem follows by checking below Lemma 10 (approximation error in W -norm) and Lemma 12 (inconsistency error). \square

We next prove the missing parts of Theorem 1. We need an estimate for interpolant I_h (see (3.5)) in the augmented norm W , and to do so, we first need to prove a trace inequality that goes from a part of boundary to the interior of the domain.

Lemma 9. Let $T \in \mathcal{T}_h^\Gamma$ and let e be an edge of T . Suppose that $w \in H^2(\omega_T^-) \cup H^2(\omega_T^+)$ and $|e^\pm| > 0$. Then, there exists a constant such that

$$\frac{1}{|e^\pm|} \|w\|_{L^2(e^\pm)}^2 \leq C \left(\frac{1}{h_T^2} \|w\|_{L^2(\omega_T^\pm)}^2 + \|Dw\|_{L^2(\omega_T^\pm)}^2 + h_T^2 \|D^2w\|_{L^2(T^\pm)}^2 \right).$$

Proof. Consider the case where $e \cap \Gamma \neq \emptyset$. Let e_1 be an interior edge of ω_T , contained in ω_T^- and connected by one node to e^- . Edge e_1 exists thanks to the r -tubular neighborhood assumption with $2h < r$. Note that $ch_T \leq |e_1|$ for a constant c that only depends on the shape regularity of the mesh. Using the fundamental theorem of calculus we can show (see Lemma 3 of [10] for a similar two dimensional result)

$$\frac{1}{|e^-|} \|w\|_{L^2(e^-)}^2 \leq C (|e^-| \|Dw\|_{L^2(e^-)}^2 + \frac{1}{h_T} \|w\|_{L^2(e_1)}^2 + h_T \|Dw\|_{L^2(e_1)}^2).$$

Noting that $|\omega_T^-| \geq Ch_T^2$, and using standard trace inequalities gives the result. The exact same argument would apply for e^+ . If $e \cap \Gamma = \emptyset$, $e^- = e$ we apply trace inequality from e to ω_T^- . \square

Lemma 10. (*Best Approximation Error Estimate*) It holds,

$$\|I_h u - u\|_W \leq C h(1 + \kappa) \left(\sqrt{\rho^-} (\|Du\|_{L^2(\Omega^-)} + \|D^2u\|_{L^2(\Omega^-)}) + \sqrt{\rho^+} (\|Du\|_{L^2(\Omega^+)} + \|D^2u\|_{L^2(\Omega^+)}) \right).$$

Here the constant C depends on the constant C_E from (3.3).

Proof. We bound each term of $\|u - I_h u\|_W$, see (4.5) and (4.1), separately. Using Lemma 5 we easily have

$$\begin{aligned} \|\sqrt{\rho} \nabla_h(u - I_h u)\|_{L^2(\Omega)}^2 &\leq C \sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma} h_T^2 \left(\rho^- \|D^2 u_E^-\|_{L^2(T)}^2 + \rho^+ \|D^2 u_E^+\|_{L^2(T)}^2 \right) \\ &+ C \sum_{T \in \mathcal{T}_h^\Gamma} h_T^2 (1 + \kappa)^2 \left(\rho^- (\|Du_E^-\|_{L^2(\omega_T)}^2 + \|D^2 u_E^-\|_{L^2(\omega_T)}^2) + \rho^+ (\|Du_E^+\|_{L^2(\omega_T)}^2 + \|D^2 u_E^+\|_{L^2(\omega_T)}^2) \right) \\ &\leq C h^2 (1 + \kappa)^2 \left(\rho^- (\|Du\|_{L^2(\Omega^-)}^2 + \|D^2 u\|_{L^2(\Omega^-)}^2) + \rho^+ (\|Du\|_{L^2(\Omega^+)}^2 + \|D^2 u\|_{L^2(\Omega^+)}^2) \right). \end{aligned}$$

In the last inequality we used (3.3). We also have used finite overlapping of the sets ω_T . We next estimate the term

$$\sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\rho^-}{|e^-|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^-)}^2 + \frac{\rho^+}{|e^+|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^+)}^2 \right).$$

Suppose that $e = \text{Int}(\partial T_1 \cap \partial T_2)$. Then, using Lemma 9 we have

$$\begin{aligned} &\frac{\rho^-}{|e^-|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^-)}^2 + \frac{\rho^+}{|e^+|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^+)}^2 \\ &\leq C \frac{\rho^-}{|e^-|} (\|u - I_{T_1} u\|_{L^2(e^-)}^2 + \|u - I_{T_2} u\|_{L^2(e^-)}^2) + C \frac{\rho^+}{|e^+|} (\|u - I_{T_1} u\|_{L^2(e^+)}^2 + \|u - I_{T_2} u\|_{L^2(e^+)}^2) \\ &\leq C \rho^- \sum_{i=1}^2 \left(\frac{1}{h_{T_i}^2} \|u - I_{T_i} u\|_{L^2(\omega_{T_i}^-)}^2 + \|D(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^-)}^2 + h_{T_i}^2 \|D^2(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^-)}^2 \right) \\ &\quad + C \rho^+ \sum_{i=1}^2 \left(\frac{1}{h_{T_i}^2} \|u - I_{T_i} u\|_{L^2(\omega_{T_i}^+)}^2 + \|D(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^+)}^2 + h_{T_i}^2 \|D^2(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^+)}^2 \right). \end{aligned}$$

If we now apply Lemma 5 and use (3.3), we get

$$\begin{aligned} &\sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\rho^-}{|e^-|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^-)}^2 + \frac{\rho^+}{|e^+|} \|\llbracket u - I_h u \rrbracket\|_{L^2(e^+)}^2 \right) \\ &\leq C h^2 (1 + \kappa)^2 \left(\rho^- (\|Du\|_{L^2(\Omega^-)}^2 + \|D^2 u\|_{L^2(\Omega^-)}^2) + \rho^+ (\|Du\|_{L^2(\Omega^+)}^2 + \|D^2 u\|_{L^2(\Omega^+)}^2) \right). \end{aligned}$$

We next bound

$$\sum_{e \in \mathcal{E}_h^\Gamma} \left(\rho^- |e^-| \|\llbracket D(u - I_h u) \rrbracket\|_{L^2(e^-)}^2 + \rho^+ |e^+| \|\llbracket D(u - I_h u) \rrbracket\|_{L^2(e^+)}^2 \right).$$

Using a trace inequality to obtain

$$\begin{aligned}
& \rho^- |e^-| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^-)}^2 + \rho^+ |e^+| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^+)}^2 \\
& \leq \rho^- |e| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^-)}^2 + \rho^+ |e| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^+)}^2 \\
& \leq \rho^- |e| \left(\|D(u - I_{T_1} u)\|_{L^2(e^-)}^2 + \|D(u - I_{T_2} u)\|_{L^2(e^-)}^2 \right) \\
& \quad + \rho^+ |e| \left(\|D(u - I_{T_1} u)\|_{L^2(e^+)}^2 + \|D(u - I_{T_2} u)\|_{L^2(e^+)}^2 \right) \\
& \leq C \rho^- \sum_{i=1}^2 \left(\|D(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^-)}^2 + h_{T_i}^2 \|D^2(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^-)}^2 \right) \\
& \quad + C \rho^+ \sum_{i=1}^2 \left(\|D(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^+)}^2 + h_{T_i}^2 \|D^2(u - I_{T_i} u)\|_{L^2(\omega_{T_i}^+)}^2 \right).
\end{aligned}$$

Again, if we apply Lemma 5 and (3.3) we obtain

$$\begin{aligned}
& \sum_{e \in \mathcal{E}_h^\Gamma} \left(\rho^- |e^-| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^-)}^2 + \rho^+ |e^+| \| \llbracket \nabla_h(u - I_h u) \rrbracket \|_{L^2(e^+)}^2 \right) \\
& \leq Ch^2(1 + \kappa)^2 \left(\rho^- (\|Du\|_{L^2(\Omega^-)}^2 + \|D^2u\|_{L^2(\Omega^-)}^2) + \rho^+ (\|Du\|_{L^2(\Omega^+)}^2 + \|D^2u\|_{L^2(\Omega^+)}^2) \right).
\end{aligned}$$

Finally, the last term is estimated by

$$\begin{aligned}
& \sum_{T \in \mathcal{E}_h} (|e^-| \rho^- \| \{ \nabla_h(u - I_h u) \} \cdot \mathbf{n} \|_{L^2(e^-)}^2 + |e^+| \rho^+ \| \{ \nabla_h(u - I_h u) \} \cdot \mathbf{n} \|_{L^2(e^+)}^2) \\
& \leq Ch^2(1 + \kappa)^2 \left(\rho^- (\|Du\|_{L^2(\Omega^-)}^2 + \|D^2u\|_{L^2(\Omega^-)}^2) + \rho^+ (\|Du\|_{L^2(\Omega^+)}^2 + \|D^2u\|_{L^2(\Omega^+)}^2) \right),
\end{aligned}$$

exactly in the same way as it was done above. \square

In order to bound the inconsistency term we need a technical lemma.

Lemma 11. *Let $T \in \mathcal{T}_h^\Gamma$ and enumerate the three edges of T by e_1, e_2, e_3 . Then, there exists a constant m such that*

$$|T_\Gamma| \leq m \max_{i=1,2,3} |e_i^-| \quad \text{and} \quad |T_\Gamma| \leq m \max_{i=1,2,3} |e_i^+|.$$

The constant m is independent of r (see Lemma 1), depends only on the shape regularity of the triangulation.

Proof. See Appendix A.2. \square

Now we are able to establish the inconsistency error estimate. The constant will be independent of the contrast of the coefficients ρ^- and ρ^+ .

Lemma 12. (Inconsistency Error Estimate) *Let u be the solution to (1.1) and $u_h \in V_h$ be its finite element approximation solution to (2.7). Then for any $d_h \in V_h$, it holds*

$$(5.4) \quad a_h(u - u_h, d_h) \leq Ch\kappa \left(\sqrt{\rho^-} (\|Du\|_{L^2(\Omega^-)} + h\|D^2u\|_{L^2(\Omega^-)}) + \sqrt{\rho^+} h\|D^2u\|_{L^2(\Omega^+)} \right) \|d_h\|_V.$$

The constant C depends on the constant C_E from (3.3).

Proof. First note that $a_h(u - u_h, d_h) = a_h(u, d_h) - (f, d_h)$. The inconsistency term is then given by

$$a_h(u - u_h, d_h) = \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\rho^- D_{\mathbf{n}^-} u^-)[d_h],$$

where we have used that $\rho^- D_{\mathbf{n}^-} u^- = \rho^+ D_{\mathbf{n}^+} u^+$ on Γ . We also have used the notation $[d_h] = d_h^+ - d_h^-$. Let L_T be the tangent line to Γ at x_0 and note that $[d_h]$ is linear function on L_T that vanishes x_0 as well as its first derivative. Hence, $[d_h] \equiv 0$ on the line L_T . Since the distance between this line and the curve T_Γ is $O(s^2)$, where we set $s = |T_\Gamma|$, we have that $|[d_h](x)| \leq C \|\mathbf{X}''\|_{L^\infty} s^2 |D[d_h]|$ for every $x \in T_\Gamma$. Hence, we have

$$\int_{T_\Gamma} (\rho^- D_{\mathbf{n}^-} u^-)[d_h] \leq C \kappa s^2 \|\sqrt{\rho^-} D_{\mathbf{n}^-} u_E^-\|_{L^2(T_\Gamma)} \sqrt{\rho^-} \|D[d_h]\|_{L^2(T_\Gamma)}.$$

For the moment, let us assume that following inequality holds

$$(5.5) \quad \sqrt{\rho^-} \sqrt{s} \|D[d_h]\|_{L^2(T_\Gamma)} \leq C M_T,$$

where

$$\begin{aligned} M_T = & \sum_{e \subset \partial T} \left(\sqrt{|e^+|} \|\sqrt{\rho^+} [\nabla_h d_h]\|_{L^2(e^+)} + \frac{1}{\sqrt{|e^+|}} \|\sqrt{\rho^+} d_h\|_{L^2(e^+)} + \|\sqrt{\rho^+} \nabla_h d_h\|_{L^2(\omega_T^+)} \right) \\ & + \sum_{e \subset \partial T} \left(\sqrt{|e^-|} \|\sqrt{\rho^-} [\nabla_h d_h]\|_{L^2(e^-)} + \frac{1}{\sqrt{|e^-|}} \|\sqrt{\rho^-} d_h\|_{L^2(e^-)} + \|\sqrt{\rho^-} \nabla_h d_h\|_{L^2(\omega_T^-)} \right). \end{aligned}$$

Then,

$$\int_{T_\Gamma} (\rho^- D_{\mathbf{n}^-} u^-)[d_h] \leq C \kappa s^{3/2} \|\sqrt{\rho^-} D u_E^-\|_{L^2(T_\Gamma)} M_T.$$

Letting B_s be the ball of radius s centered at x_0 we can use the trace inequality to get

$$\|\sqrt{\rho^-} D u_E^-\|_{L^2(T_\Gamma)} \leq C \left(\frac{1}{\sqrt{s}} \|\sqrt{\rho^-} D u_E^-\|_{L^2(B_s)} + \sqrt{s} \|\sqrt{\rho^-} D^2 u_E^-\|_{L^2(B_s)} \right).$$

Hence,

$$\int_{T_\Gamma} (\rho^- D_{\mathbf{n}^-} u^-)[d_h] \leq C s (\|\sqrt{\rho^-} D u_E^-\|_{L^2(B_s)} + s \|\sqrt{\rho^-} D^2 u_E^-\|_{L^2(B_s)}) M_T.$$

We see that (5.4) follows after using that

$$\sum_{T \in \mathcal{T}_h^\Gamma} M_T^2 \leq C \|d_h\|_V^2,$$

the inequality (3.3) and the fact that $s \leq C h_T$ (see Lemma A.1).

In order to complete the proof we need to prove (5.5). Firstly, by using the triangle inequality we have

$$\sqrt{\rho^-} \sqrt{s} \|D[d_h]\|_{L^2(T_\Gamma)} \leq \sqrt{\rho^-} \sqrt{s} (\|D d_h^-\|_{L^2(T_\Gamma)} + \|D d_h^+\|_{L^2(T_\Gamma)}).$$

Then by Lemma 11, there exists an edge e of T such that

$$(5.6) \quad s \leq m |e^+|.$$

Now let $e = \text{Int}(\partial T \cap \partial K)$, for a $K \in \mathcal{T}_h$. Using that $D d_h^+$ is constant we get

$$\|D d_h^+\|_{L^2(T_\Gamma)} = \sqrt{s} |D d_h^+| = \frac{\sqrt{s}}{\sqrt{|e^+|}} \|D d_h^+\|_{L^2(e^+)} \leq \sqrt{m} \|D d_h^+\|_{L^2(e^+)}.$$

According to Lemma 6

$$(5.7) \quad |e^+|^2 \leq \theta \max\{|T^+|, |K^+|\}.$$

First, suppose that $|T^+| = \max\{|T^+|, |K^+|\}$. Then,

$$\|\nabla_h d_h^+\|_{L^2(e^+)} = \frac{\sqrt{|e^+|}}{\sqrt{|T^+|}} \|Dd_h^+\|_{L^2(T^+)}.$$

Hence,

$$\|\nabla_h d_h^+\|_{L^2(e^+)} \leq \frac{\sqrt{\theta m}}{\sqrt{s}} \|Dd_h^+\|_{L^2(T^+)}.$$

On the other hand, if $|K^+| = \max\{|T^+|, |K^+|\}$ then we have

$$\|\nabla_h d_h^+\|_{L^2(e^+)} \leq \|D(d_h^+|_T - d_h^+|_K)\|_{L^2(e^+)} + \frac{\sqrt{\theta m}}{\sqrt{s}} \|Dd_h^+\|_{L^2(K^+)}.$$

We write $D(d_h|_T - d_h|_K)$ as its normal part and tangential part, then, use an inverse inequality for the tangential part on e^+ , and have

$$\|\nabla_h(d_h^+|_T - d_h^+|_K)\|_{L^2(e^+)} \leq \|[\![\nabla_h d_h]\!]\|_{L^2(e^+)} + \frac{C}{|e^+|} \|[\![d_h]\!]\|_{L^2(e^+)},$$

therefore,

$$\|\nabla_h d_h^+\|_{L^2(e^+)} \leq \|[\![\nabla_h d_h]\!]\|_{L^2(e^+)} + \frac{\sqrt{\theta m}}{\sqrt{s}} \|Dd_h^+\|_{L^2(K^+)} + \frac{C}{|e^+|} \|[\![d_h]\!]\|_{L^2(e^+)}.$$

Combining the above inequalities we obtain

$$\begin{aligned} \sqrt{\rho^- s} \|Dd_h^+\|_{L^2(T_\Gamma)} &\leq \sqrt{m|e^+|} \sqrt{\rho^+} \|[\![\nabla_h d_h]\!]\|_{L^2(e^+)} + \frac{C \sqrt{\rho^+} \sqrt{m}}{\sqrt{|e^+|}} \|[\![d_h]\!]\|_{L^2(e^+)} \\ &\quad + Cm\sqrt{\theta} (\|\sqrt{\rho^+} Dd_h\|_{L^2(T^+)} + \|\sqrt{\rho^+} Dd_h\|_{L^2(K^+)}), \end{aligned}$$

where we have used $\rho^- \leq \rho^+$, so

$$\begin{aligned} \sqrt{\rho^- s} \|Dd_h^+\|_{L^2(T_\Gamma)} &\leq C \sum_{e \in \partial T} (\sqrt{m|e^+|} \sqrt{\rho^+} \|[\![\nabla_h d_h]\!]\|_{L^2(e^+)} + \frac{\sqrt{\rho^+} \sqrt{m}}{\sqrt{|e^+|}} \|[\![d_h]\!]\|_{L^2(e^+)}) \\ &\quad + m\sqrt{\theta} \|\sqrt{\rho^+} \nabla_h d_h\|_{L^2(\omega_T^+)}. \end{aligned}$$

Using the same argument we can prove

$$\begin{aligned} (\rho^- s)^{1/2} \|Dd_h^-\|_{L^2(T_\Gamma)} &\leq C \sum_{e \in \partial T} (\sqrt{m|e^-|} \sqrt{\rho^-} \|[\![Dd_h]\!]\|_{L^2(e^-)} + \frac{\sqrt{\rho^-} \sqrt{m}}{\sqrt{|e^-|}} \|[\![d_h]\!]\|_{L^2(e^-)}) \\ &\quad + Cm\sqrt{\theta} \|\sqrt{\rho^-} \nabla_h d_h\|_{L^2(\omega_T^-)}. \end{aligned}$$

Combining the two last inequalities we obtain (5.5). \square

Now we would like to state a corollary of Theorem 1. We first need some regularity results. We start with a standard energy estimate.

Proposition 1. *Let u solve (1.1), then the following energy estimate holds*

$$(5.8) \quad \sqrt{\rho^+} \|Du\|_{L^2(\Omega^+)} + \sqrt{\rho^-} \|Du\|_{L^2(\Omega^-)} \leq \frac{C}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}$$

where C is independent of ρ^\pm .

At this point we recall that we are assuming that $\rho^- \leq \rho^+$. In fact, a better energy estimate holds as we prove in the next proposition.

Proposition 2. *Let u solve (1.1), then the following energy estimate holds*

$$(5.9) \quad \rho^+ \|Du\|_{L^2(\Omega^+)} + \rho^- \|Du\|_{L^2(\Omega^-)} \leq C \|f\|_{L^2(\Omega)}.$$

The constant C could depend on the geometry of the subdomains Ω^\pm .

Proof. We prove this estimate in the case that Ω^+ is the inclusion (i.e. $\partial\Omega^+$ does not intersect $\partial\Omega$). The proof of the other case is similar. First, from the previous proposition we have

$$(5.10) \quad \rho^- \|Du\|_{L^2(\Omega^-)} \leq C \|f\|_{L^2(\Omega)}.$$

Let $w = u^+ - \frac{1}{|\Omega^+|} \int_{\Omega^+} u^+ dx$. Therefore, using Poincaré's inequality we have

$$(5.11) \quad \|w\|_{L^2(\Omega^+)} \leq C \|Du\|_{L^2(\Omega^+)}.$$

By means of an extension result (see Lemma 6.37 in [11]), there exists $v \in H_0^1(\Omega)$ such that $v|_{\Omega^+} = w|_{\Omega^+}$ such that

$$(5.12) \quad \|v\|_{H^1(\Omega)} \leq C \|w\|_{H^1(\Omega^+)}.$$

Applying the variational formulation of problem (1.1) we have that

$$\int_{\Omega^-} \rho^- Du \cdot Dv + \int_{\Omega^+} \rho^+ Du \cdot Dv dx = \int_{\Omega} f v.$$

Multiplying by ρ^+ it follows that

$$\|\rho^+ Du\|_{L^2(\Omega^+)}^2 = \rho^+ \int_{\Omega^+} \rho^+ Du \cdot Dv dx = \rho^+ \int_{\Omega} f v - \rho^+ \int_{\Omega^-} \rho^- Du \cdot Dv$$

Therefore, we have

$$\|\rho^+ Du\|_{L^2(\Omega^+)}^2 \leq \|f\|_{L^2(\Omega)} \|\rho^+ v\|_{L^2(\Omega)} + \|\rho^- Du\|_{L^2(\Omega^-)} \|\rho^+ Dv\|_{L^2(\Omega)}.$$

By (5.12) we have

$$(5.13) \quad \|\rho^+ Du\|_{L^2(\Omega^+)}^2 \leq C(\|f\|_{L^2(\Omega)} + \|\rho^- Du\|_{L^2(\Omega^-)}) \rho^+ \|w\|_{H^1(\Omega^+)}.$$

Hence, using (5.11) we get

$$(5.14) \quad \|\rho^+ Du\|_{L^2(\Omega^+)}^2 \leq C(\|f\|_{L^2(\Omega)} + \|\rho^- Du\|_{L^2(\Omega^-)}) \|\rho^+ Du\|_{L^2(\Omega^+)}.$$

The proof is complete after applying (5.10). \square

We will also need to state and H^2 regularity estimates which can essentially be proved if one combines results in [8] and [19]. Here we give an alternative argument using the results in [19] and classical regularity theory.

Proposition 3. *In addition to the assumptions already made in this article, assume further that Ω is convex. Let u solve (1.1). Then, the following regularity estimate holds*

$$(5.15) \quad \rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)} \leq C_{reg} \|f\|_{L^2(\Omega)},$$

where C is independent of ρ^\pm .

The constant C_{reg} could depend on the geometry of the subdomains Ω^\pm .

Proof. We consider two cases:

Case 1: Ω^- is the inclusion (i.e. $\partial\Omega^- \cap \partial\Omega = \emptyset$)

In this case, (5.15) follows from (2.35) in [19].

Case 2: Ω^+ is the inclusion

From (2.36) in [19] we have

$$(5.16) \quad \rho^- \|u\|_{H^2(\Omega^-)} \leq C \|f\|_{L^2(\Omega)}.$$

Let $\bar{u}^+ = \frac{1}{|\Omega^+|} \int_{\Omega^+} u^+ dx$. Let $w = u^+ - \bar{u}^+$. Note that w satisfies

$$\begin{aligned} -\rho^+ \Delta w &= f && \text{in } \Omega^+, \\ \rho^+ \nabla w \cdot \mathbf{n} &= \rho^+ \nabla u^+ \cdot \mathbf{n} && \text{on } \partial\Omega^+. \end{aligned}$$

From a standard regularity result (see (2.3.3.1) in [14]) we have

$$\rho^+ \|w\|_{H^2(\Omega^+)} \leq C(\|f\|_{L^2(\Omega^+)} + \|\rho^+ \nabla u^+ \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega^+)} + \rho^+ \|w\|_{H^1(\Omega^+)}).$$

Using the jump condition we have

$$\|\rho^+ \nabla u^+ \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega^+)} = \|\rho^- \nabla u^- \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega^+)}.$$

By a standard trace inequality we have

$$\|\rho^- \nabla u^- \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega^+)} \leq \rho^- \|u^-\|_{H^2(\Omega^-)}.$$

Which by (5.16) gives

$$\|\rho^+ \nabla u^+ \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega^+)} \leq C \|f\|_{L^2(\Omega)}.$$

The estimate

$$\rho^+ \|w\|_{H^1(\Omega^+)} \leq C \|f\|_{L^2(\Omega)}$$

follows from Poincare's inequality and Proposition 2. Hence, we have

$$\rho^+ \|w\|_{H^2(\Omega^+)} \leq C \|f\|_{L^2(\Omega)}.$$

The result now follows after noting that $\|D^2 u\|_{L^2(\Omega^+)} \leq \|w\|_{H^2(\Omega^+)}.$ \square

Combining Theorem 1 and the previous propositions we have the following corollary.

Corollary 1. *Let u be the solution to (1.1) and $u_h \in V_h$ be its finite element approximation. If Ω is convex, we have*

$$(5.17) \quad \|u - u_h\|_V \leq \frac{C h(1 + \kappa)}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}.$$

The constant C depends on the constant C_E from (3.3) and the constant C_{reg} from (5.15).

5.2. L^2 -Error Estimates. We now prove an L^2 error estimate using a duality argument.

Theorem 2. *Let u be the solution to problem (1.1) and $u_h \in V_h$ be its finite element approximation solution to (2.7). Assume that Ω is convex. Then, there exists a constant $C > 0$ independent of h , ρ^- and ρ^+ , such that*

$$(5.18) \quad \|u - u_h\|_{L^2(\Omega)} \leq \frac{C h^2(1 + \kappa)^2}{\rho^-} \|f\|_{L^2(\Omega)}.$$

The constant C depends on C_E from (3.3) and C_{reg} from (5.15).

Proof. Let ϕ be a solution of the problem

$$\begin{aligned} (5.19a) \quad & -\rho^\pm \Delta \phi^\pm = (u - u_h)^\pm && \text{in } \Omega^\pm, \\ (5.19b) \quad & \phi = 0 && \text{on } \partial\Omega, \\ (5.19c) \quad & [\phi] = 0 && \text{on } \Gamma, \\ (5.19d) \quad & [\rho D_{\mathbf{n}} \phi] = 0 && \text{on } \Gamma. \end{aligned}$$

We have

$$\|u - u_h\|_{L^2(\Omega)}^2 = \int_{\Omega} (u - u_h)u - \int_{\Omega} (u - u_h)u_h = a_h(\phi, u) - a_h(\phi_h, u_h),$$

where ϕ_h is the finite element approximation of ϕ . Therefore, we see that

$$\begin{aligned} (5.20) \quad \|u - u_h\|_{L^2(\Omega)}^2 &= a_h(\phi, u - u_h) + a_h(\phi - \phi_h, u_h) \\ &= a_h(\phi - \phi_h, u - u_h) + a_h(u - u_h, \phi_h) + a_h(\phi - \phi_h, u_h) \\ &= a_h(\phi - \phi_h, u - u_h) + a_h(u - u_h, \phi_h - I_h \phi) + a_h(u - u_h, I_h \phi) \\ &\quad + a_h(\phi - \phi_h, u_h - I_h u) + a_h(\phi - \phi_h, I_h u). \end{aligned}$$

Using continuity of the bilinear form we have

$$a_h(\phi - \phi_h, u - u_h) \leq C \|\phi - \phi_h\|_W \|u - u_h\|_W.$$

Using the triangle inequality we have

$$\|u - u_h\|_W \leq \|u - I_h u\|_W + \|I_h u - u_h\|_W.$$

It is not difficult to show that

$$\|I_h u - u_h\|_W \leq C \|I_h u - u_h\|_V.$$

Hence,

$$\|u - u_h\|_W \leq \|u - I_h u\|_W + \|u - u_h\|_V.$$

Now using Theorem 10, Theorem 1, (5.15) and (5.8) we get

$$\|u - u_h\|_W \leq \frac{C h(1 + \kappa)}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}.$$

Similarly,

$$\|\phi - \phi_h\|_W \leq \frac{C h(1 + \kappa)}{\sqrt{\rho^-}} \|u - u_h\|_{L^2(\Omega)},$$

and hence we have a bound for the first term in (5.20)

$$a_h(\phi - \phi_h, u - u_h) \leq \frac{C h^2(1 + \kappa)^2}{\rho^-} \|f\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

Using Lemma 12, (5.15) and (5.8) we have

$$a_h(u - u_h, \phi_h - I_h \phi) \leq \frac{C h(1 + \kappa)}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)} \|I_h \phi - \phi_h\|_V,$$

which implies the following bound for the second term in (5.20)

$$a_h(u - u_h, \phi_h - I_h \phi) \leq \frac{C h^2(1 + \kappa)^2}{\rho^-} \|f\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

Analogously, for the fourth term in (5.20) we have

$$a_h(\phi - \phi_h, u_h - I_h u) \leq \frac{C h^2 (1 + \kappa)^2}{\rho^-} \|f\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

For the third term in (5.20) we have

$$(5.21) \quad a_h(u - u_h, I_h \phi) = \sum_{T \in \mathcal{T}_h^\Gamma} \int_{T_\Gamma} (\rho^- D_{\mathbf{n}^-} u^-) [I_h \phi].$$

Using the Cauchy-Schwarz inequality we obtain

$$a_h(u - u_h, I_h \phi) \leq \|\sqrt{\rho^-} D_{\mathbf{n}^-} u^-\|_{L^2(\Gamma)} \sqrt{\rho^-} \left(\sum_{T \in \mathcal{T}_h^\Gamma} \| [I_h \phi] \|_{L^2(T_\Gamma)}^2 \right)^{1/2},$$

For the first term in (5.2) we apply a trace inequality to obtain

$$\|\sqrt{\rho^-} D_{\mathbf{n}^-} u^-\|_{L^2(\Gamma)} \leq C \sqrt{\rho^-} (\|D^2 u\|_{L^2(\Omega^-)} + \|Du\|_{L^2(\Omega^-)}) \leq \frac{C}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}.$$

where we used (5.15), (5.8) and the fact that $\rho^- \leq \rho^+$. Now for the second term in (5.2), we note that $[I_h \phi] = 0$ on L_T and that L_T is at most distance $\mathcal{O}(s^2)$ from T_Γ where $s = |T_\Gamma|$. Therefore we can use, Taylor's theorem to show that

$$[I_h \phi](x) \leq C \|\mathbf{X}''\|_{L^\infty} s^2 |D[I_h \phi]|(x) \quad \forall x \in T_\Gamma.$$

Hence,

$$\|[I_h \phi]\|_{L^2(T_\Gamma)} \leq C \kappa r^2 \|D[I_h \phi]\|_{L^2(T_\Gamma)} \leq C \kappa h_T^2 \|D[I_h \phi]\|_{L^2(T_\Gamma)},$$

where we used that $s \leq C h_T$. Consequently, adding and subtracting $D[\phi]$

$$\left(\sum_{T \in \mathcal{T}_h^\Gamma} \|[I_h \phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2} \leq C \kappa \left(\sum_{T \in \mathcal{T}_h^\Gamma} h_T^4 \|D[I_h \phi - \phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2} + C \kappa \left(\sum_{T \in \mathcal{T}_h^\Gamma} h_T^4 \|D[\phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2}.$$

Observe that

$$h^2 \|D[\phi]\|_{L^2(\Gamma)} \leq C h^2 (\|D\phi^+\|_{L^2(\Gamma)} + \|D\phi^-\|_{L^2(\Gamma)}).$$

Application of trace inequality gives

$$\left(\sum_{T \in \mathcal{T}_h^\Gamma} \|D[\phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2} \leq C h^2 (\|D^2 \phi\|_{L^2(\Omega^-)} + \|D\phi\|_{L^2(\Omega^-)} + \|D^2 \phi\|_{L^2(\Omega^+)} + \|D\phi\|_{L^2(\Omega^+)}).$$

For the other term we use that $I_h \phi|_T = I_T \phi$ and use a trace inequality to bound

$$\begin{aligned} \|D[I_h \phi - \phi]\|_{L^2(T_\Gamma)} &= \|D[I_T \phi - \phi]\|_{L^2(T_\Gamma)} \\ &\leq \frac{C}{\sqrt{h_T}} (\|D(I_T \phi - \phi)\|_{L^2(\omega_T^-)} + \|D(I_T \phi - \phi)\|_{L^2(\omega_T^+)}) \\ &\quad + C \sqrt{h_T} (\|D^2 \phi\|_{L^2(\omega_T^-)} + \|D^2 \phi\|_{L^2(\omega_T^+)}). \end{aligned}$$

From (3.6) and (3.7) (using that $\rho^- \leq \rho^+$) we obtain

$$\|D(I_T \phi - \phi)\|_{L^2(\omega_T^-)} + \|D(I_T \phi - \phi)\|_{L^2(\omega_T^+)} \leq C h_T (\|D^2 \phi\|_{L^2(\omega_T^-)} + \|D^2 \phi\|_{L^2(\omega_T^+)}).$$

Therefore,

$$\left(\sum_{T \in \mathcal{T}_h^\Gamma} h_T^4 \|D[I_h \phi - \phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2} \leq C h^2 \kappa (\|D^2 \phi\|_{L^2(\Omega^-)} + \|D\phi\|_{L^2(\Omega^-)}) \\ + C h^2 (\|D^2 \phi\|_{L^2(\Omega^+)} + \|D\phi\|_{L^2(\Omega^+)}).$$

Hence, using the regularity result (5.15) and (5.8) we have

$$\sqrt{\rho^-} \left(\sum_{T \in \mathcal{T}_h^\Gamma} \|[I_h \phi]\|_{L^2(T_\Gamma)}^2 \right)^{1/2} \leq \frac{C h^2 \kappa}{\sqrt{\rho^-}} \|u - u_h\|_{L^2(\Omega)}.$$

Thus, we obtain the bound for the third term in (5.20)

$$a_h(u - u_h, I_h \phi) \leq \frac{C h^2 \kappa}{\rho^-} \|f\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

In a similar fashion we can prove

$$a_h(\phi - \phi_h, I_h u) \leq \frac{C h^2 \kappa}{\rho^-} \|f\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

The proof is complete after combining the above inequalities for the terms in (5.20). \square

6. EXTENSIONS AND FINAL REMARKS

6.1. Extension to three dimensions. We now show that the space $S^1(T)$, introduced in (2.4), can easily be extended in three-dimensions. We consider the problem (1.1) where Ω is a three-dimensional domain and Γ is a simple, closed, C^2 surface. Let now \mathcal{T}_h be a simplicial triangulation of Ω . Let \mathcal{E}^h be all the faces of the mesh \mathcal{T}_h . Finally, let \mathcal{E}_h^Γ be the set of all faces that belong to a tetrahedron that intersects Γ .

We now define the local finite element space. Let $T \in \mathcal{T}_h$ be a tetrahedron. Let x_0 be a fixed point on $\Gamma \cap T$. Let \mathbf{n}_0^+ be the outward pointing unit vector normal to Ω^+ at x_0 . Let $\mathbf{t}_0^+, \mathbf{s}_0^+$ be such that $\{\mathbf{n}_0^+, \mathbf{t}_0^+, \mathbf{s}_0^+\}$ forms an orthonormal system.

Given $v \in \mathbb{P}^1(T^+)$ there exists a unique $\Upsilon(v) \in \mathbb{P}^1(T^-)$ satisfying

$$\begin{aligned} \Upsilon(v)(x_0) &= v(x_0), \\ (D_{\mathbf{t}_0^+} \Upsilon(v))(x_0) &= (D_{\mathbf{t}_0^+} v)(x_0), \\ (D_{\mathbf{s}_0^+} \Upsilon(v))(x_0) &= (D_{\mathbf{s}_0^+} v)(x_0), \\ \rho^- (D_{\mathbf{n}_0^+} \Upsilon(v))(x_0) &= \rho^+ (D_{\mathbf{n}_0^+} v)(x_0). \end{aligned}$$

Given $T \in \mathcal{T}_h^\Gamma$ and for each $v \in \mathbb{P}^1(T^+)$ we can consider the unique corresponding function

$$G(v) = \begin{cases} v & \text{in } T^+, \\ \Upsilon(v) & \text{in } T^-. \end{cases}$$

Let $\text{span}\{v_1, v_2, v_3, v_4\}$ be a basis for $\mathbb{P}^1(T)$ restricted to T^+ . Then we define the local finite element space

$$S^1(T) = \begin{cases} \text{span}\{G(v_1), G(v_2), G(v_3), G(v_4)\} & , \text{ if } T \in \mathcal{T}_h^\Gamma \\ \mathbb{P}^1(T) & , \text{ if } T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma. \end{cases}$$

Consequently, the global finite element space is given by

$$V_h := \{v : v|_T \in S^1(T), \forall T \in \mathcal{T}_h, v \text{ is continuous across all faces in } \mathcal{E}_h \setminus \mathcal{E}_h^\Gamma\}.$$

We believe that (2.7) can be extend and analyzed to the three-dimensional case, where now $e^\pm = e \cap \Omega^\pm$ are pieces of a face e of a tetrahedra, and considering appropriate stabilization parameters. This could be a subject for a future work.

6.2. Alternative Local Spaces. An alternative approach to enforce weak continuity of our local space $S^1(T)$ is normally used in the literature (see [8]). Instead of enforcing continuity of function and tangential derivative at x_0 one imposes continuity at two distinct points x_1, x_2 . More precisely, define x_1, x_2 be the two points in Γ that intersect ∂T . Then, one can define $\Upsilon(v)$ (in contrast to the definition in Lemma 2) by

$$\begin{aligned} \Upsilon(v)(x_i) &= v(x_i) \quad \text{for } i = 1, 2 \\ \rho^-(D_{\mathbf{n}_0^+} \Upsilon(v))(x_0) &= \rho^+(D_{\mathbf{n}_0^+} v)(x_0). \end{aligned}$$

Subsequently, $I_T u$ is now defined by

$$\begin{aligned} (I_T^- u)(x_i) &:= (J_T u_E^+)(x_i) =: (I_T^+ u)(x_i) \quad \text{for } i = 1, 2 \\ \rho^-(D_{\mathbf{n}_0^+} I_T^- u)(x_0) &:= \rho^+(D_{\mathbf{n}_0^+} J_T u_E^+)(x_0) =: \rho^+(D_{\mathbf{n}_0^+} I_T^+ u)(x_0). \end{aligned}$$

Defining the finite element method using these local spaces, we can prove all the a-priori estimates above.

Another alternative is to enforce the matching conditions by averaging on T_Γ . More precisely, one can redefine $\Upsilon(v)$ in Lemma 2 by

$$\begin{aligned} \int_{T_\Gamma} \Upsilon(v) ds &:= \int_{T_\Gamma} v, \\ \int_{T_\Gamma} \Upsilon(v) s ds &:= \int_{T_\Gamma} v s, \\ \int_{T_\Gamma} \rho^+ D_{\mathbf{n}^+} \Upsilon(v) &:= \int_{T_\Gamma} \rho^- D_{\mathbf{n}^+} v, \end{aligned}$$

or alternatively, one can replace the second equation by

$$\int_{T_\Gamma} D_{\mathbf{t}^+} \Upsilon(v) ds := \int_{T_\Gamma} D_{\mathbf{t}^+} v ds.$$

Even though their numerical implementation is more complicated and require numerical integrations, these alternatives have the advantage that the analysis is shorter especially for the consistency error, and also can be formulated using Lagrange multipliers.

6.3. Extension to Cartesian grids. We note that the analysis and methods developed here can be easily extended to Cartesian grids and \mathbb{Q}^1 elements. One possibility would be to use same local spaces $S^1(T)$ defined before for quadrangular elements $T \in \mathcal{T}_h^\Gamma$, that is, one has three degrees of freedom for $T \in \mathcal{T}_h^\Gamma$ and four degrees otherwise. In case one wants to have four degrees of freedom for $T \in \mathcal{T}_h^\Gamma$, let the space $S^1(T)$ then be a piecewise bilinear function under

the coordinates $\mathbf{n}_0, \mathbf{t}_0$ and impose

$$\begin{aligned}\Upsilon(v)(x_0) &:= v(x_0) \\ (D_{\mathbf{t}_0^+} \Upsilon(v))(x_0) &:= (D_{\mathbf{t}_0^+} v)(x_0) \\ \rho^-(D_{\mathbf{n}_0^+} \Upsilon(v))(x_0) &:= \rho^+(D_{\mathbf{n}_0^+} v)(x_0) \\ \rho^-(D_{\mathbf{n}_0^+} D_{\mathbf{t}_0^+} \Upsilon(v))(x_0) &:= \rho^+(D_{\mathbf{n}_0^+} D_{\mathbf{t}_0^+} v)(x_0)\end{aligned}$$

6.4. Alternative Bilinear forms. In this section we discuss alternative bilinear forms. We will point out what are the theoretical difficulties in analyzing these alternative methods. When we provide numerical experiments in the following section we will also see that, although we cannot prove stability for some methods, they sometimes do well experimentally in some of the norms.

Firstly, formulation without flux stabilization has been used for example in [21]. Methods (6.1) and (6.2) experiment this direction.

$$\begin{aligned}(6.1) \quad a_h(w, v) &:= \int_{\Omega} \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^{\Gamma}} \int_e (\{\rho \nabla_h v\} \cdot \llbracket w \rrbracket + \{\rho \nabla_h w\} \cdot \llbracket v \rrbracket) \\ &\quad + \gamma \sum_{e \in \mathcal{E}_h^{\Gamma}} \frac{1}{|e|} \left(\int_{e^-} \rho^- \llbracket w \rrbracket \cdot \llbracket v \rrbracket + \int_{e^+} \rho^+ \llbracket w \rrbracket \cdot \llbracket v \rrbracket \right).\end{aligned}$$

$$\begin{aligned}(6.2) \quad a_h(w, v) &:= \int_{\Omega} \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^{\Gamma}} \int_e (\{\rho \nabla_h v\} \cdot \llbracket w \rrbracket + \{\rho \nabla_h w\} \cdot \llbracket v \rrbracket) \\ &\quad + \gamma \sum_{e \in \mathcal{E}_h^{\Gamma}} \left(\frac{1}{|e^-|} \int_{e^-} \rho^- \llbracket w \rrbracket \cdot \llbracket v \rrbracket + \frac{1}{|e^+|} \int_{e^+} \rho^+ \llbracket w \rrbracket \cdot \llbracket v \rrbracket \right).\end{aligned}$$

The problem with these methods is that we cannot establish neither the optimal inconsistency error nor the coercivity independent of the contrast and the mesh.

We note that methods (6.1) and (6.2) seem to do well numerically in the L^2 and energy norms, however, they are mesh dependent in the L^∞ norm and they do not converge in the weighted $W^{1,\infty}$ norm.

The method that seems to do the best numerically, slightly better than the proposed method, is the following one:

$$\begin{aligned}(6.3) \quad a_h(w, v) &:= \int_{\Omega} \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^{\Gamma}} \int_e (\{\rho \nabla_h v\} \cdot \llbracket w \rrbracket + \{\rho \nabla_h w\} \cdot \llbracket v \rrbracket) \\ &\quad + \gamma \sum_{e \in \mathcal{E}_h^{\Gamma}} \left(\frac{1}{|e^-|} \int_{e^-} \rho^- \llbracket w \rrbracket \cdot \llbracket v \rrbracket + \frac{1}{|e^+|} \int_{e^+} \rho^+ \llbracket w \rrbracket \cdot \llbracket v \rrbracket \right) \\ &\quad + \gamma_F \sum_{e \in \mathcal{E}_h^{\Gamma}} |e| \left(\int_{e^-} \rho^- \llbracket \nabla_h v \rrbracket \llbracket \nabla_h w \rrbracket + \int_{e^+} \rho^+ \llbracket \nabla_h v \rrbracket \llbracket \nabla_h w \rrbracket \right).\end{aligned}$$

The difference between this method and the one analyzed in this paper is that we are using a stronger flux stabilization (i.e. we replace $|e^\pm|$ by $|e|$ and we introduce a flux stabilization parameter γ_F). It is not difficult to see, that we can prove all the error estimates contained in this paper for this method. In particular, the coercivity of the bilinear form is obvious since we are adding even more stabilization. Clearly, now we would have to redefine our V and W norms

but the approximation properties will still hold. In summary, Theorem 1 and Corollaries 1 and 2 hold for this formulation.

A very natural question would be if we can penalize the jump terms with $1/|e|$ instead of $1/|e^\pm|$ and get a stable and optimally convergent method independent of contrast. The answer is yes, as long as we penalize the jump of the full gradient instead of the flux.

$$\begin{aligned}
 (6.4) \quad a_h(w, v) &:= \int_{\Omega} \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^\Gamma} \int_e (\{\rho \nabla_h v\} \cdot \llbracket w \rrbracket + \{\rho \nabla_h w\} \cdot \llbracket v \rrbracket) \\
 &\quad + \gamma \sum_{e \in \mathcal{E}_h^\Gamma} \frac{1}{|e|} \left(\int_{e^-} \rho^- \llbracket w \rrbracket \cdot \llbracket v \rrbracket + \int_{e^+} \rho^+ \llbracket w \rrbracket \cdot \llbracket v \rrbracket \right) \\
 &\quad + \gamma_F \sum_{e \in \mathcal{E}_h^\Gamma} |e| \left(\int_{e^-} \rho^- [\nabla v]_\otimes \cdot [\nabla w]_\otimes + \int_{e^+} \rho^+ [\nabla v]_\otimes \cdot [\nabla w]_\otimes \right),
 \end{aligned}$$

where the jumps in the last line are defined by

$$[\nabla v]_\otimes = \nabla v^- \otimes \mathbf{n}^- + \nabla v^+ \otimes \mathbf{n}^+.$$

and the symbol \otimes denotes the outer product between vectors. We note that in the coercivity and inconsistency error analysis proofs, we have used the inverse inequality $|e^-| \|D_{\mathbf{n}} v\|_{L^2(e^-)} \leq C \|\nabla v\|_{L^2(T^-)}$ (where T^- is a triangle with edge e , and similar result for e^+) in particular in the estimation of the left-hand side of (4.4). For method (6.4) we want to avoid using this inverse estimate and the factor $1/|e^-|$ (also $1/|e^+|$). Let x be the vertex of e belonging to Ω^- . Using the ideas in the proof of Lemma 6, we can find a sequence of elements $\{T_2, T_3, \dots, T_N\}$ in the patch of x , and common edges $\{e_2, e_3, \dots, e_{N-1}\}$ with $\bar{e}_i = \bar{T}_i \cap \bar{T}_{i+1}$ and $e_i^- := e_i \cap \Omega^-$, such that, $|e^-| \leq \sigma |e_2^-| \leq \sigma^2 |e_3^-| \leq \sigma^{N-2} |e_{N-1}^-| = \sigma^{N-1} |e|$ where σ a positive constant bounded uniformly from below by zero and N is bounded depending on the shape regularity of the mesh. Then, we bound the left-hand side of (4.4) by using recursively

$$\|\nabla v|_{T_{i-1}}\|_{L^2(e_i^-)}^2 \leq 2 \left(\|[\nabla v]_\otimes\|_{L^2(e_i^-)}^2 + \|\nabla v|_{T_i}\|_{L^2(e_i^-)}^2 \right),$$

and

$$\|\nabla v|_{T_i}\|_{L^2(e_i^-)}^2 \leq 1/\sigma \|\nabla v|_{T_i}\|_{L^2(e_{i+1}^-)}^2,$$

and then use that

$$\|\nabla v|_{T_N}\|_{L^2(e_{N-1}^-)}^2 \leq (C/|e|) \|\nabla v\|_{L^2(T_N^-)}^2.$$

Hence,

$$|e| \|\nabla v\|_{L^2(e^-)}^2 \leq C \sum_{i=2}^N |e| \|[\nabla v]_\otimes\|_{L^2(e_i^-)}^2 + \frac{C}{|e|} \|\nabla v\|_{L^2(T_N^-)}^2.$$

6.5. Final remarks. In this article rigorous error estimates independent of contrast were derived. However, many interesting research questions remain. Firstly, we kept track, as much as possible, of how the constants depend on the geometry (e.g. curvature). However, there are two constants, C_E and C_{reg} , that we did not investigate in detail how they scale with geometric quantities such as maximum curvature and the radius r of the tubular neighborhood. We believe it is possible to prove a bound for C_E in terms of geometric quantities, but for the sake of simplicity we did not investigate it here. In addition, the regularity constant C_{reg} appearing in 5.15 depends on the geometry. Addressing these issues will lead to error estimates

that are completely explicit on their dependence on the curvature and radius r of the tubular neighborhood.

Secondly, our method and error estimates do not consider problems with high curvature. An interesting line of research would be to investigate problems with interfaces of arbitrary curvature. In this direction, perhaps a combination of the method presented here and the multcale method by Chu et al. in [8] could be a possible approach.

Thirdly, rigorous error analysis for higher order Immersed Finite Element methods ($k > 1$) remains open. It would be interesting to study this, in particular, for high-contrast problems.

Finally, we would like to study the sharpness of our Assumptions (1)-(3). For instance, we observe that our method is still well-defined without Assumption (3).

7. NUMERICAL EXAMPLES

In this section we explore the properties of the methods presented in sections above applied to the two dimensional interface problem (1.1). In particular, we are interested in the computation of the following errors and their respective estimated order of convergence

$$\begin{aligned}
e_h^0 &:= \|u - u_h\|_{L^2(\Omega)}, & e_h^\infty &:= \|u - u_h\|_{L^\infty(\Omega)}, \\
e_h^1 &:= \|\sqrt{\rho}\nabla_h(u - u_h)\|_{L^2(\Omega)}, & e_h^{1,\infty} &:= \|\sqrt{\rho}\nabla_h(u - u_h)\|_{L^\infty(\Omega)}, \\
\bar{e}_h^1 &:= \|\rho\nabla_h(u - u_h)\|_{L^2(\Omega)}, & \bar{e}_h^{1,\infty} &:= \|\rho\nabla_h(u - u_h)\|_{L^\infty(\Omega)}, \\
e_h^{n,\infty} &:= \|\rho D_{\mathbf{n}}(u - u_h)\|_{L^\infty(\Gamma)}, & \tilde{e}_h^{1,\infty} &:= \|\rho\nabla_h(u - u_h)\|_{L^\infty(\Omega \setminus (\cup\{T: T \in \mathcal{T}_h^\Gamma\}))} \\
\text{e.o.c.} &:= \frac{\log(e_{h_{l+1}}/e_{h_l})}{\log(h_{l+1}/h_l)}.
\end{aligned}$$

Computation (or approximation) of the L^∞ norms are performed evaluating the error at the following set of points: if an element does not intersect the interface then the set of points are the nodes and the centroid of the element, if the element does intersect the interface we evaluate at the nodes and at the two points intersections of the interface with the edges of the element. We observe that the errors corresponding to the L^2 norm and $W^{1,2}$ weighted with $\sqrt{\rho}$ semi-norm correspond to the only results proved in this paper, Theorem 2 and 1 respectively. We expect optimal convergence, second order in the L^2 norm and first order in the weighted $W^{1,2}$ semi-norm. We also compute the analogous in the L^∞ norm and $W^{1,\infty}$ weighted with $\sqrt{\rho}$ semi-norm. In addition, we compute the errors in the $W^{1,2}$ and $W^{1,\infty}$ semi-norm weighted with ρ . Note that the estimate for the interpolation error is also achieved in this semi-norm. Indeed, this is a consequence of Lemma 5 and Proposition 3 and 2

$$\|\rho\nabla_h(u - I_h u)\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

A revealing result of our experiments is that the ratio of convergence of the error is optimal when we compute using the triangles non intersecting the interface. Error $\tilde{e}_h^{1,\infty}$ illustrates this observation. Finally, error $e_h^{n,\infty}$ is a standard error for interface problems, illustrating the approximation of the normal derivative on the interface.

The experiment presented below shows that method (2.7)-(6.3) produces the best results. This method (and all the others) does not seem optimal for the $W^{1,\infty}$ error weighted with ρ . However, it is optimal when we do not consider the elements in \mathcal{T}_h^Γ . We highlight that this result was not remotely addressed in this paper, and this kind of estimates appear to be more difficult. For the method that we analyzed (2.7)-(2.6) we observe an uncertain behavior in the error $e_h^{1,\infty}$ for the last mesh, not achieving the optimal convergence as in the previous method. We also present tables for one of the method without flux stabilization, method (2.7)-(6.2). We

observe non convergence so far for the error $e_h^{1,\infty}$ and in the last mesh and a deterioration in the L^∞ norm. We also do not have the optimal convergence for the normal flux error $e_h^{n,\infty}$.

In our numerical experiment we consider the two dimensional domain $\Omega = (-1, 1)^2$ with the immersed interface $\Gamma = \{\mathbf{x} \in \Omega : \mathbf{x}_1^2 + \mathbf{x}_2^2 = R^2\}$. We define $\Omega^- := \{\mathbf{x} \in \Omega : \mathbf{x}_1^2 + \mathbf{x}_2^2 < R^2\}$ and $\Omega^+ = \Omega \setminus (\Omega^- \cup \Gamma)$. Example (1) considers the following exact solution

$$u(\mathbf{x}) = \begin{cases} \frac{R^\alpha}{\rho^-} & , \text{ if } \mathbf{x} \in \Omega^-, \\ \frac{R^\alpha}{\rho^+} + \frac{1}{3^\alpha} \left(\frac{1}{\rho^-} - \frac{1}{\rho^+} \right) & , \text{ if } \mathbf{x} \in \Omega^+, \end{cases}$$

where $R = \sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2}$ and $\alpha = 2$. For example (1) we have $\Gamma = \partial\Omega^-$. Similar results were obtained for the case $\Gamma = \partial\Omega^+$. We provide plots of the approximate solution for both cases.

Finite element uniform triangular meshes \mathcal{T}_h non matching the interface were used. In the tables we compute with $h = 2^{-(l+3/2)}$, for $l = 1, \dots, 7$.

All the computations were performed in MATLAB, including solution of the linear system by means of “\”.

- (1) Case $\Gamma = \partial\Omega^-$, $\rho^+ = 10^4$ and $\rho^- = 1$. Tables 1 and 2 show the results obtained by method (2.7)-(6.3) with stabilization parameters $\gamma = 10$ and $\gamma_F = 10$. Note that this method has a stronger flux stabilization than the method analyzed in the paper.

l	e_h^0	e.o.c.	e_h^∞	e.o.c.	e_h^1	e.o.c.	$e_h^{1,\infty}$	e.o.c.
1	8.2e-3		2.5e-2		1.1e-1		3.7e-1	
2	1.7e-3	2.28	5.7e-3	2.12	4.4e-2	1.30	2.1e-1	0.86
3	2.7e-4	2.63	1.3e-3	2.19	1.8e-2	1.29	9.7e-2	1.08
4	4.6e-5	2.57	3.2e-4	1.97	8.3e-3	1.12	5.2e-2	0.90
5	9.0e-6	2.34	7.2e-5	2.15	3.9e-3	1.07	2.5e-2	1.08
6	2.0e-6	2.19	1.8e-5	2.01	1.9e-3	1.03	1.3e-2	0.92
7	4.7e-7	2.08	4.6e-6	1.94	9.5e-4	1.02	6.8e-3	0.94

l	\bar{e}_h^1	e.o.c.	$\bar{e}_h^{1,\infty}$	e.o.c.	$\tilde{e}_h^{1,\infty}$	e.o.c.	$e_h^{n,\infty}$	e.o.c.
1	3.9e-1		7.0e-1		7.0e-1		3.7e-1	
2	1.6e-1	1.32	6.1e-1	0.19	4.0e-1	0.81	2.1e-1	0.86
3	6.4e-2	1.29	1.9e-1	1.68	1.9e-1	1.07	9.7e-2	1.08
4	2.9e-2	1.15	2.2e-1	-0.20	1.0e-1	0.91	4.9e-2	1.00
5	1.4e-2	1.09	1.4e-1	0.65	5.0e-2	1.01	2.5e-2	0.98
6	6.6e-3	1.04	6.6e-2	1.09	2.5e-2	0.99	1.2e-2	1.00
7	3.2e-3	1.02	5.1e-2	-2.95	1.3e-2	0.98	6.2e-3	1.00

TABLE 1. Example (1); errors and convergence orders with $\rho^- = 1$ and $\rho^+ = 10^4$, using method (2.7)-(6.3) with stabilization parameters $\gamma = 10$ and $\gamma_F = 10$.

The results in Table 1 show optimal convergence for the errors e_h^1 and e_h^0 validating the theoretical results Theorem 1 and 2. In addition we observe optimal convergence for the error e_h^∞ and $e_h^{1,\infty}$. The error \bar{e}_h^1 , weighted with ρ instead of $\sqrt{\rho}$, converges optimally. However, the rate of convergence of the error $\bar{e}_h^{1,\infty}$ is not optimal. An interesting observation is that the error $\tilde{e}_h^{1,\infty}$ converges optimally, indicating that is only a couple of

ρ^+	e_h^0	$\bar{e}_h^{1,\infty}$	e_h^1
10^1	2.3e-6	6.5e-3	2.8e-3
10^2	2.0e-6	6.6e-3	2.0e-3
10^3	2.0e-6	6.6e-3	1.9e-3
10^4	2.0e-6	6.6e-3	1.9e-3
10^5	2.0e-6	6.6e-3	1.9e-3
10^6	2.0e-6	6.6e-3	1.9e-3

TABLE 2. Example (1); errors with $\rho^- = 1$ and $h = 2^{-(6+3/2)}$, using method (2.7)-(6.3) with stabilization parameters $\gamma = 10$ and $\gamma_F = 10$.

elements where the error does not converge. Another appealing feature of the method is the optimal order of convergence for the error $e_h^{n,\infty}$.

Table 2 shows that the errors are independent of the contrast. We can observe that although we increase ρ^+ the errors remain constant, showing the contrast independency of our estimates.

We present in Table 3 the errors and convergence orders for the method analyzed in the paper (2.7)-(2.6).

l	e_h^0	e.o.c.	e_h^∞	e.o.c.	e_h^1	e.o.c.	$e_h^{1,\infty}$	e.o.c.
1	8.6e-3		2.6e-2		1.1e-1		3.7e-1	
2	1.7e-3	2.31	6.0e-3	2.14	4.5e-2	1.30	2.0e-1	0.90
3	2.8e-4	2.63	1.3e-3	2.18	1.8e-2	1.31	9.3e-2	1.09
4	4.7e-5	2.57	3.4e-4	1.97	8.4e-3	1.12	5.5e-2	0.76
5	9.2e-6	2.35	7.6e-5	2.16	4.0e-3	1.08	2.6e-2	1.09
6	2.0e-6	2.20	1.9e-5	2.01	1.9e-3	1.03	1.4e-2	0.91
7	4.7e-7	2.09	5.1e-6	1.88	9.5e-4	1.02	7.1e-2	-2.37

l	\bar{e}_h^1	e.o.c.	$\bar{e}_h^{1,\infty}$	e.o.c.	$\tilde{e}_h^{1,\infty}$	e.o.c.	$e_h^{n,\infty}$	e.o.c.
1	4.0e-1		7.4e-1		7.4e-1		3.7e-1	
2	1.6e-1	1.33	2.5e+0	-1.79	4.0e-1	0.90	2.0e-1	0.90
3	6.4e-2	1.31	2.8e-1	3.20	1.9e-1	1.07	9.2e-2	1.11
4	2.9e-2	1.15	1.6e+0	-2.56	9.7e-2	0.96	4.5e-2	1.01
5	1.4e-2	1.09	4.8e-1	1.76	4.7e-2	1.04	2.3e-2	0.96
6	6.6e-3	1.05	4.8e-1	0.01	2.4e-2	1.00	1.1e-2	1.05
7	3.2e-3	1.02	7.1e+0	-3.88	1.8e-2	0.42	5.9e-3	0.93

TABLE 3. Example (1); errors and convergence orders with $\rho^- = 1$ and $\rho^+ = 10^4$, using method (2.7)-(2.6) with stabilization parameters $\gamma = 10$.

The results in Table 3 show optimal convergence for the errors e_h^1 and e_h^0 validating the theoretical results Theorem 1 and 2. In addition we observe optimal convergence for the error e_h^∞ . The error $e_h^{1,\infty}$ seems to converge optimal up to mesh $l = 6$, and then for the last mesh the rate can possibly be affected by the choice of the flux stabilization

parameter. As in the previous test the error \bar{e}_h^1 converges optimally, however for the rest of the errors the convergence is not as clear as in the previous method. We suspect that this phenomena is related to the weights $|e^\pm|$ in the flux stabilization.

Finally, Table 4 displays error and convergence orders for the method without flux stabilization (2.7)-(6.2).

l	e_h^0	e.o.c.	e_h^∞	e.o.c.	e_h^1	e.o.c.	$e_h^{1,\infty}$	e.o.c.
1	1.7e-3		7.0e-3		5.8e-2		2.2e-1	
2	4.5e-3	1.88	2.9e-3	1.29	3.1e-2	0.89	1.6e-1	0.51
3	3.6e-4	0.31	6.1e-3	-1.08	1.2e-1	-1.89	1.3e+1	-6.37
4	2.8e-5	3.69	2.1e-4	4.87	7.7e-3	3.92	2.0e-1	5.98
5	7.3e-6	1.96	5.0e-5	2.05	3.8e-3	1.02	1.5e-1	0.40
6	1.7e-6	2.06	1.2e-5	2.05	1.9e-3	1.01	2.5e-2	-0.70
7	4.5e-7	1.95	4.0e-6	1.60	9.5e-4	0.99	1.1e-1	0.23

l	\bar{e}_h^1	e.o.c.	$\bar{e}_h^{1,\infty}$	e.o.c.	$\bar{e}_h^{1,\infty}$	e.o.c.	$e_h^{n,\infty}$	e.o.c.
1	2.1e-1		5.8e-1		2.2e-1		1.5e-1	
2	1.3e-1	0.69	1.4e+1	-4.58	1.5e-1	0.49	1.1e-1	0.39
3	7.3e-1	-2.46	1.3e+2	0.10	8.9e-1	-2.5	1.3e+1	-6.83
4	2.8e-2	4.70	7.2e+0	0.84	5.3e-2	4.07	2.0e-1	6.00
5	1.3e-2	1.07	2.6e+0	1.50	2.9e-2	0.86	1.5e-1	0.39
6	7.0e-3	0.93	2.1e+0	0.27	3.7e-2	-0.35	2.5e-1	-0.71
7	3.3e-3	0.99	8.6e+0	-2.10	2.1e-2	0.86	1.1e-3	0.23

TABLE 4. Example (1); errors and convergence orders with $\rho^- = 1$ and $\rho^+ = 10^4$, using method (2.7)-(6.2) with stabilization parameters $\gamma = 10$.

The results in Table 4 show optimal asymptotical convergence for the errors e_h^1 and e_h^0 . In addition we observe a slightly sub-optimal convergence for the error e_h^∞ , approximately 1.8. The error $e_h^{1,\infty}$ does not seem to converge. As in the previous test the error \bar{e}_h^1 converges asymptotically to 1, however for the rest of the errors we do not observe convergence.

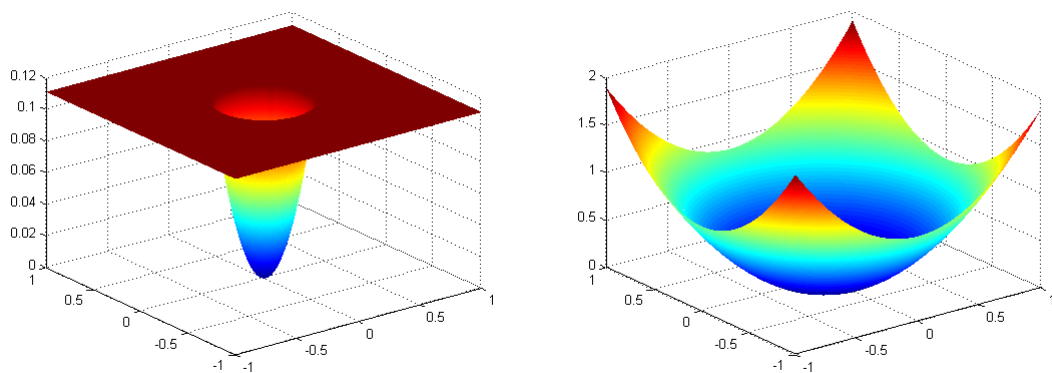


FIGURE 3. Approximate solution Example (1) (left) case $\Gamma = \partial\Omega^-$ and for case $\Gamma = \partial\Omega^+$ (right).

APPENDIX A. TECHNICAL LEMMAS

In this section we prove two technical lemmas involving geometrical estimates on elements intersected by the interface Γ . We remind the reader that we assume that Γ is a simple \mathcal{C}^2 curve with an arc-length parameterization $\mathbf{X} : [0, |\Gamma|) \rightarrow \Gamma$, and assumption (1)-(3) in Section 2.1.

We assume that r is the radius of our tubular neighborhood given in Lemma 1. Equivalently, for any $0 < \tau \leq r$ we have

$$Tub(\tau) = \{x : \text{dist}(x, \Gamma) \leq \tau\}.$$

Moreover, for any $x \in Tub(r)$ there exists a unique $x_\Gamma \in \Gamma$ such that $\text{dist}(x, x_\Gamma) = \text{dist}(x, \Gamma)$ and $x - x_\Gamma$ is perpendicular to Γ at x_Γ .

Lemma A.1. *Consider the Γ_s a segment of the curve Γ and ℓ the straight segment connecting the two end points of Γ_s . Assume that $|\ell| \leq r$, where r is the radius of the r -tubular neighborhood. Then, it holds*

$$|\Gamma_s| \leq 2|\ell|.$$

Proof. Let x_1, x_2 be the endpoints of the line segment ℓ . For $i = 1, 2$, let N_i be the line that is normal to Γ at x_i . Let \tilde{S} be the infinite region enclosed by N_1 and N_2 , then let $S = Tub(r/2) \cap \tilde{S}$ be the tubular section of Γ_s . We see that the area of S is given by $|S| = r|\Gamma_s|$. Let M_i $i = 1, 2$ be the two lines that are parallel to ℓ and distance r from ℓ . Consider the trapezoid, R , enclosed by $N_i, M_i, i = 1, 2$. We note that the area of R is given by $|R| = 2r|\ell|$. The result will follow if we show that $S \subset R$ since this will imply that $r|\Gamma_s| \leq 2r|\ell|$.

To this end, we first note that the line segment $\ell \subset Tub(r/2)$ since the distance of any point in ℓ to Γ is less than $|\ell|/2$ which we are assuming is less than $r/2$. Let $x \in S$, then we know there exists a unique point $x_\Gamma \in \Gamma_s$ where the distance from x to x_Γ is less than $r/2$ and the line, which we denote by L_x , that passes through x and x_Γ is perpendicular to Γ at x_Γ . We know that L_x intersects ℓ and we call this point y . We also know that distance between y and x_Γ is less than $r/2$ since $y \in \ell \subset Tub(r/2)$. Hence, we have shown that $\text{dist}(y, x) \leq r$. This, of course implies that the distance of x to the infinite line ℓ_{ext} (which is the line that contains the line segment ℓ) is at most r . In other words, x is in between the two lines M_1 and M_2 . Since x was in the tubular section S it was in between the two lines N_1 and N_2 and hence x belongs to the trapezoid R . \square

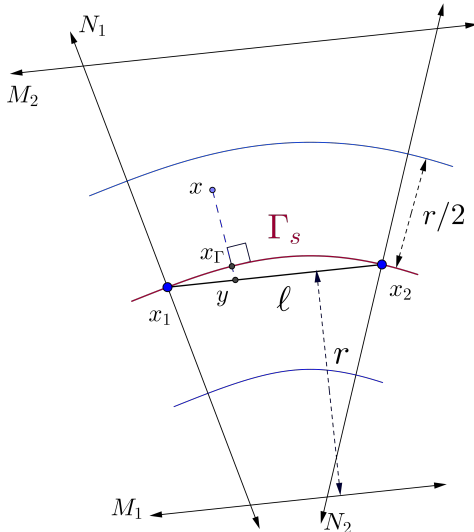


FIGURE 4. Illustration of a curve segment Γ_s , straight line segment ℓ connecting the end points of Γ_s , the tubular section of radius S , and the trapezoid R .

A.1. Proof of Lemma 6. Let $e = \text{Int}(\partial T_1 \cap \partial T_2) \in \mathcal{E}_h^\Gamma$, with $T_1, T_2 \in \mathcal{T}_h$, and the previous definitions of $e^\pm = e \cap \Omega^\pm$. We analyze the case in Ω^- . The same analysis is valid in Ω^+ . We proceed analyzing two cases depending on the intersection of the edge e with Γ :

- (i) If $e \cap \Gamma = \emptyset$, ($e = e^-$). If either T_1 or T_2 does not belong to \mathcal{T}_h^Γ the results is trivial. Consider the case where T_1 and T_2 belong to \mathcal{T}_h^Γ , the interface sections $T_{1,\Gamma}$ and $T_{2,\Gamma}$ are nonempty. We consider the midpoint m_e the segment e and the ball $B_{|e|/2}(m_e)$ of radius $|e|/2$ and centered at m_e . If the interface does not cross the ball $B_{|e|/2}(m_e)$ then we have

$$|T_1^-| \geq |B_{|e|/2}(m_e) \cap T_1|.$$

Denote by $\underline{\alpha}$ the minimum angle of the triangulation (given by shape regularity), and let $\tilde{\alpha} = \min\{\underline{\alpha}, \pi/4\}$. Now consider the isosceles triangle \tilde{T}_1 with base edge e and base angles $\tilde{\alpha}$. Then, clearly $\tilde{T}_1 \subset B_{|e|/2}(m_e) \cap T_1$ and $|\tilde{T}_1| = (|e^-|/2)^2 \tan\{\tilde{\alpha}\}$. Therefore, $|T_1^-| \geq (|e^-|/2)^2 \tan\{\tilde{\alpha}\}$. Assume then that the interface crosses the ball $B_{|e|/2}(m_e)$ in T_1 . Observe that this implies that there exists a point in $T_{1,\Gamma}$ who's normal passes through m_e with distance less than $|e|/2$. Now, if $T_{2,\Gamma}$ crosses the ball then we will have two points on Γ at a distance less than $|e|/2$ whose normal passes through m_e which contradicts the tubular neighborhood assumption. Therefore

$$\max_{i=\{1,2\}} |T_i^-| \geq \left(\frac{|e|}{2}\right)^2 \tan\{\tilde{\alpha}\}.$$

- (ii) If $e \cap \Gamma \neq \emptyset$. Similarly as in the previous case, consider m_e the midpoint of e^- and the ball of radius $|e^-|/2$ centered at m_e . We observe that this ball can not cross both $T_{1,\Gamma}$ and $T_{2,\Gamma}$. Therefore

$$\max_{i=\{1,2\}} |T_i^-| \geq \left(\frac{|e^-|}{2}\right)^2 \tan\{\tilde{\alpha}\}.$$

A.2. Proof of Lemma 11. We analyze the case in Ω^- . We first observe that, by triangle inequality we have

$$|l_T| \leq \sum_{i=1}^3 |e_i^-| \leq 3 \max_{i=\{1,2,3\}} |e_i^-|.$$

Therefore, using Lemma A.1

$$|T_\Gamma| \leq 6 \max_{i=1,2,3} |e_i^-|.$$

Same analysis is valid to prove the statement in Ω^+ .

APPENDIX B. CONSTRUCTION OF BASIS FUNCTIONS OF $S^1(T)$

In this section we define basis functions $\{w_1, w_2, w_3\}$ of the space $S^1(T)$, for $T \in \mathcal{T}_h^\Gamma$. Consider $\{x_1, x_2, x_3\}$ nodes of the element T . Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the barycentric coordinates of a point $x \in T$ with respect to $\{x_1, x_2, x_3\}$. Consider the following representation

$$w_i(x) = \begin{cases} w_i^-(x), & \text{if } x \in T^-; \\ w_i^+(x), & \text{if } x \in T^+. \end{cases} \quad w_i^\pm(x) = \sum_{j=1}^3 a_{i,j}^\pm \lambda_j(x).$$

We construct $\{w_1, w_2, w_3\}$, a set of basis functions of the space $S^1(T)$, by satisfying the following conditions: for $i = 1, 2, 3$

$$\begin{cases} w_i(x_j) = \delta_{i,j} & \text{for } j = 1, 2, 3; \\ [w_i(x_0)] = 0 & x_0 \in T_\Gamma; \\ [D_{\mathbf{t}_0} w_i] = 0 & \mathbf{t}_0 = \mathbf{t}(x_0); \\ [D_{\mathbf{n}_0} w_i] = 0 & \mathbf{n}_0 = \mathbf{n}(x_0); \end{cases} \quad \text{where } \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

These conditions are written in a system of size 6×6 using the barycentric coordinates representation of w_i^\pm , i.e., we find the unknowns coefficients $\{a_{i,j}^\pm\}_{j=1}^3$ of w_i for $i = 1, 2, 3$, solutions of:

$$\begin{pmatrix} \tilde{\delta}_{1,+} & 0 & 0 & \tilde{\delta}_{1,-} & 0 & 0 \\ 0 & \tilde{\delta}_{2,+} & 0 & 0 & \tilde{\delta}_{2,-} & 0 \\ 0 & 0 & \tilde{\delta}_{3,+} & 0 & 0 & \tilde{\delta}_{3,-} \\ \lambda_1(x_0) & \lambda_2(x_0) & \lambda_3(x_0) & -\lambda_1(x_0) & -\lambda_2(x_0) & -\lambda_3(x_0) \\ -D_{\mathbf{t}_0} \lambda_1 & -D_{\mathbf{t}_0} \lambda_2 & -D_{\mathbf{t}_0} \lambda_3 & D_{\mathbf{t}_0} \lambda_1 & D_{\mathbf{t}_0} \lambda_2 & D_{\mathbf{t}_0} \lambda_3 \\ -\rho^+ D_{\mathbf{n}_0} \lambda_1 & -\rho^+ D_{\mathbf{n}_0} \lambda_2 & \rho^+ D_{\mathbf{n}_0} \lambda_3 & \rho^- D_{\mathbf{n}_0} \lambda_1 & \rho^- D_{\mathbf{n}_0} \lambda_2 & \rho^- D_{\mathbf{n}_0} \lambda_3 \end{pmatrix} \begin{pmatrix} a_{i,1}^+ \\ a_{i,2}^+ \\ a_{i,3}^+ \\ a_{i,1}^- \\ a_{i,2}^- \\ a_{i,3}^- \end{pmatrix} = \begin{pmatrix} \delta_{i,1} \\ \delta_{i,2} \\ \delta_{i,3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where

$$\tilde{\delta}_{j,\pm} = \begin{cases} 1, & \text{if } x_j \in T^\pm \\ 0, & \text{if } x_j \in T^\mp. \end{cases}$$

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